

Holographic Entanglement of Purification

Koji Umemoto (YITP, Kyoto U.)

w/ Tadashi Takayanagi (YITP) Nat. Phys. 14, 573 (2018).

Yang Zhou (Fudan U.) arXiv:1805.02625.

- **Bekenstein-Hawking entropy**

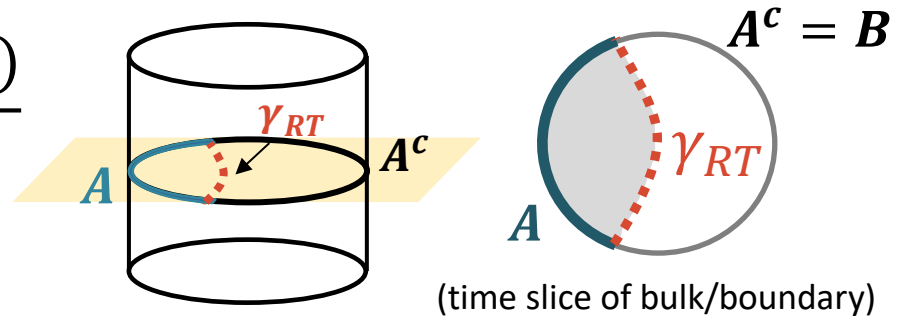
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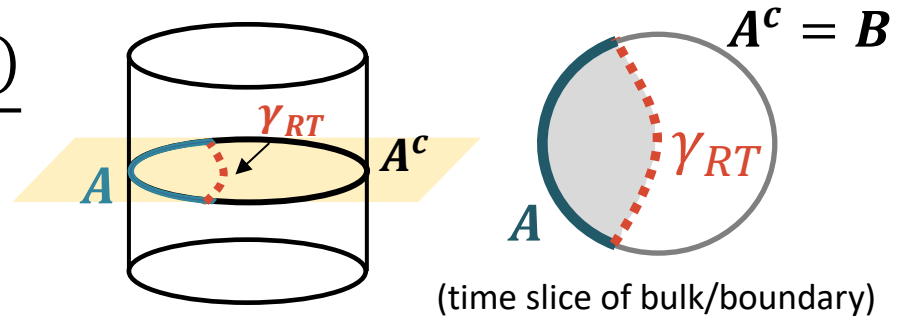
[Refer to Xi Dong's Review Talk]

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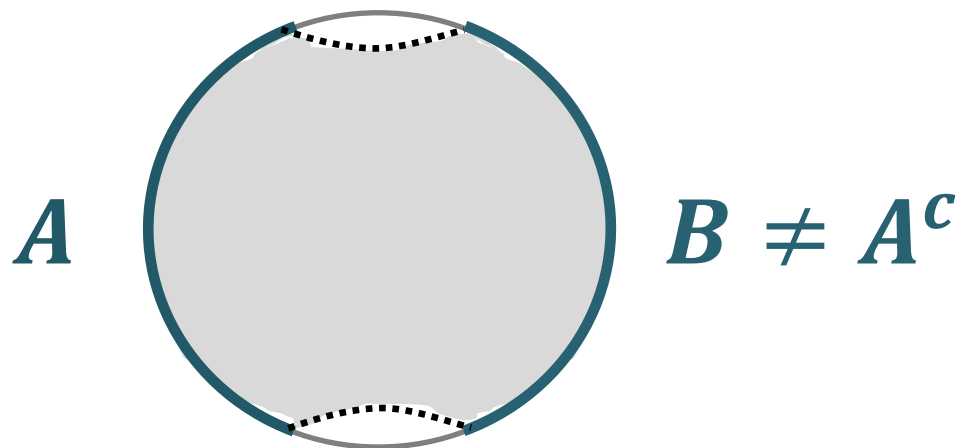
- **Entanglement \cong Geometry ?**

Tensor Network, ER=EPR, Bit Threads,...

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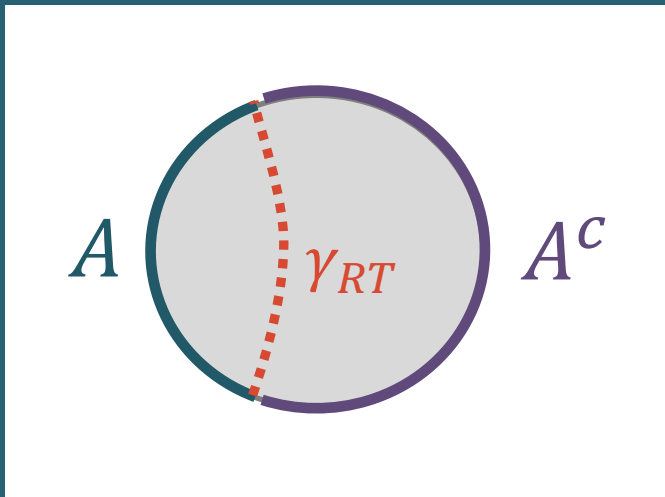


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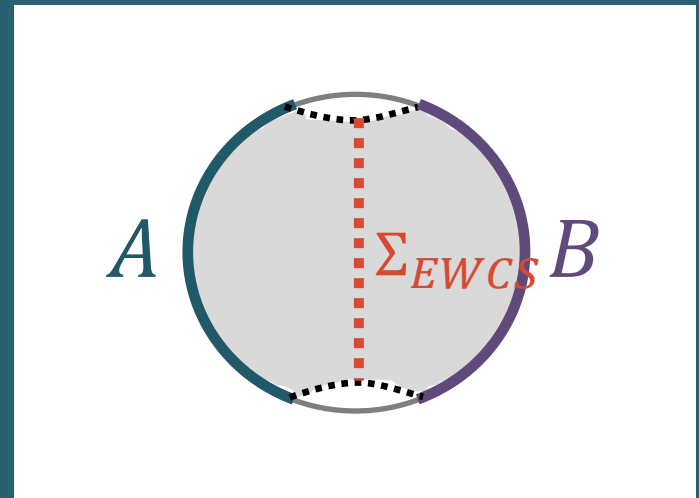
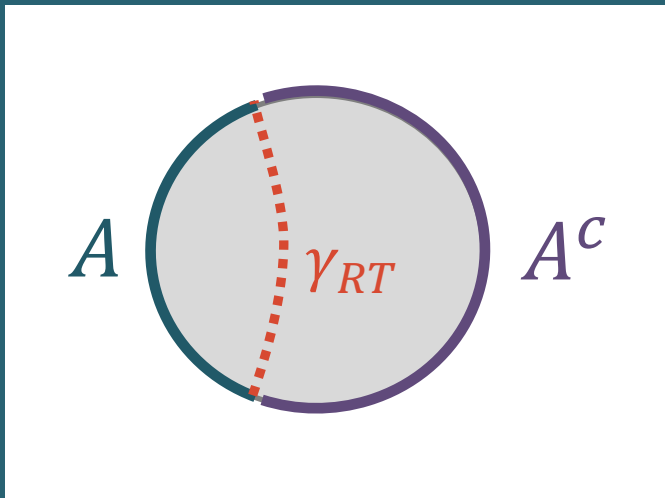
Question

Entanglement \cong Geometry for **mixed state setups**?

Entanglement entropy



Entanglement entropy \rightarrow *Entanglement of purification*



- **Entanglement of purification** [*Terhal-Horodecki-Leung-DiVincenzo '02*]

$$E_P(\rho_{AB}) := \min_{|\Psi\rangle_{AA'BB'}} S_{AA'} \quad (\rho_{AA'} := \text{Tr}_{BB'}[|\Psi\rangle\langle\Psi|_{AA'BB'}])$$

- A measure of **total correlation** (entanglement + classical one)

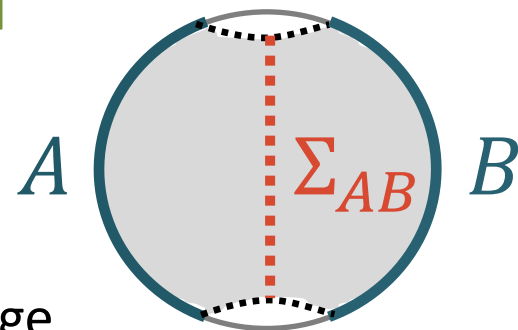
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- **Holographic dual: Entanglement wedge cross-section**

[Takayanagi-KU, Nguyen-Devakul-Halbasch-Zaletel-Swingle '17]

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- Minimal cross-section of entanglement wedge

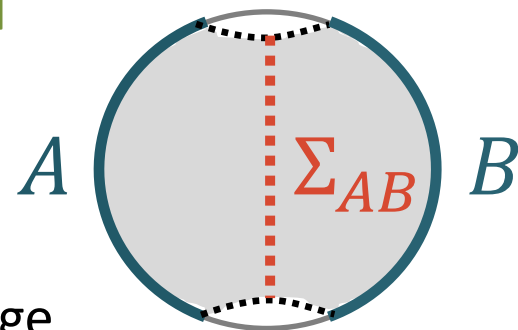
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- **Conjecture**

$$E_P = E_W$$

- **Agreements of information-theoretic properties**

- ✓ For pure states: $E_{\#}(|\psi\rangle_{AB}) = S_A$
- ✓ Faithfulness: $E_{\#}(\rho_{AB}) = 0$ if and only if $\rho_{AB} = \rho_A \otimes \rho_B$
- ✓ Less than entanglement entropies: $E_{\#}(\rho_{AB}) \leq \min[S_A, S_B]$
- ✓ Never increases by trace out: $E_{\#}(\rho_{AB_1B_2}) \geq E_{\#}(\rho_{AB_1})$
- ✓ $E_{\#}(\rho_{AB}) \geq I(A:B)/2$, $E_{\#}(\rho_{AB_1B_2}) \geq I(A:B_1)/2 + I(A:B_2)/2 \dots$

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- **Tensor network description**

$$E_P(\rho_{AB}) = \min_{|\psi\rangle_{AA'BB'}} \left(\text{Diagram 1} \right) = \left(\text{Diagram 2} \right) = E_W(\rho_{AB})$$

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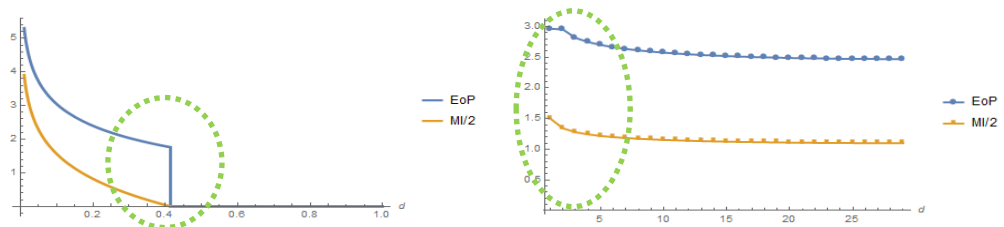
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The diagram shows two circular tensor networks representing the same quantity. The first diagram shows a circle with a vertical dashed red line representing a cut, with labels A and B on the left and right, and A' and B' at the top and bottom. A red label $S_{AA'}$ is placed near the cut. The second diagram is identical but with a blue arc at the top and bottom, and a red label $E_W(\rho_{AB})$ to the right.

- **Numerical results in toy models**

Ex. Phase Transition of E_W/E_P

[Bhattacharyya-Takayanagi-KU '18]



- **Multipartite entanglement of purification** [*Bao-Halpern, Zhou-KU '18*]

$$\Delta_W(\rho_{ABC}) := \frac{1}{2} \min_{|\psi\rangle_{AA'BB'CC'}} [S_{AA'} + S_{BB'} + S_{CC'}]$$

- New measure of multipartite total correlation
- We proved various information-theoretic properties of Δ_P

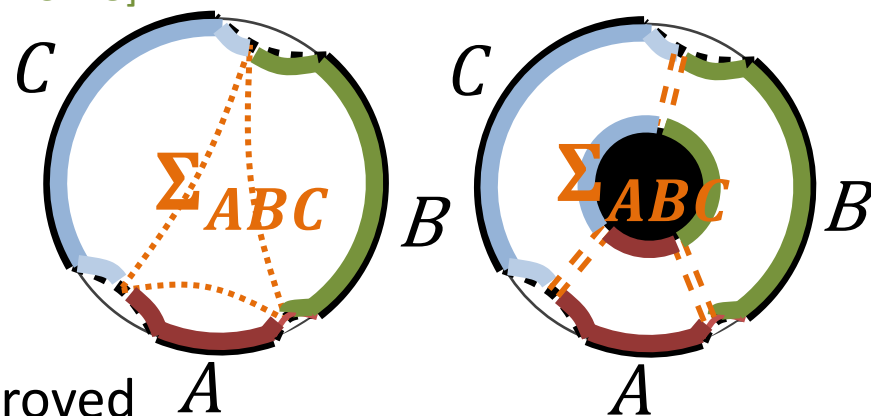
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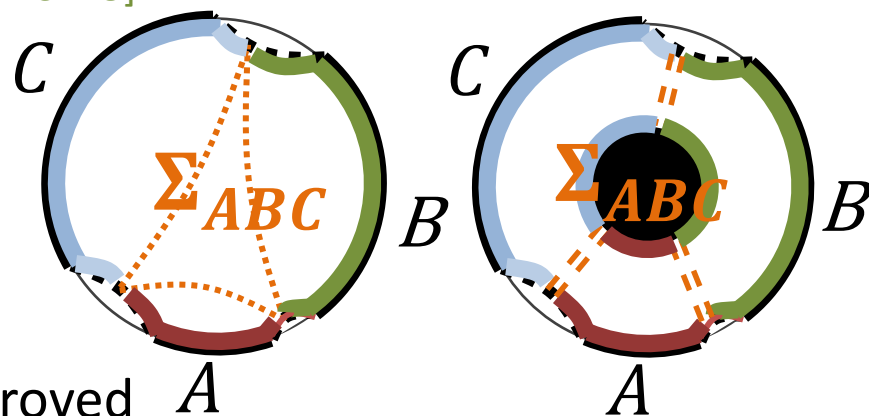
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- **Generalized conjecture**

$$\Delta_P = \Delta_W$$

- **Squashed entanglement in holography**
 - Monogamy of holographic mutual information [*Hayden-Headrick-Maloney '11*]
 - **Multipartite strong superadditivity of multipartite holographic MI**
- **Operational meanings of E_P**
 - EPR pairs and LOq - Tensor network? Bit threads?
- **Conditional entanglement of purification** [*Bao-Halpern '17, '18*]
- **Strong superadditivity of E_W**
- **Future directions**
 - CFT calculation of E_P [*Hirai-Tamaoka-Yokoya '18*]
 - Operational interpretation of Δ_P
 - Structure of quantum states saturating $E_{Sq} \leq I/2 \dots$

Thank you for your attention