Black Holes and Random Matrices II: New Universality at Early Time?

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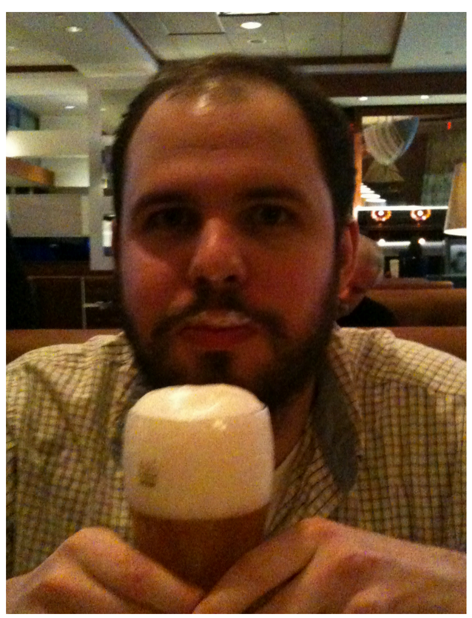

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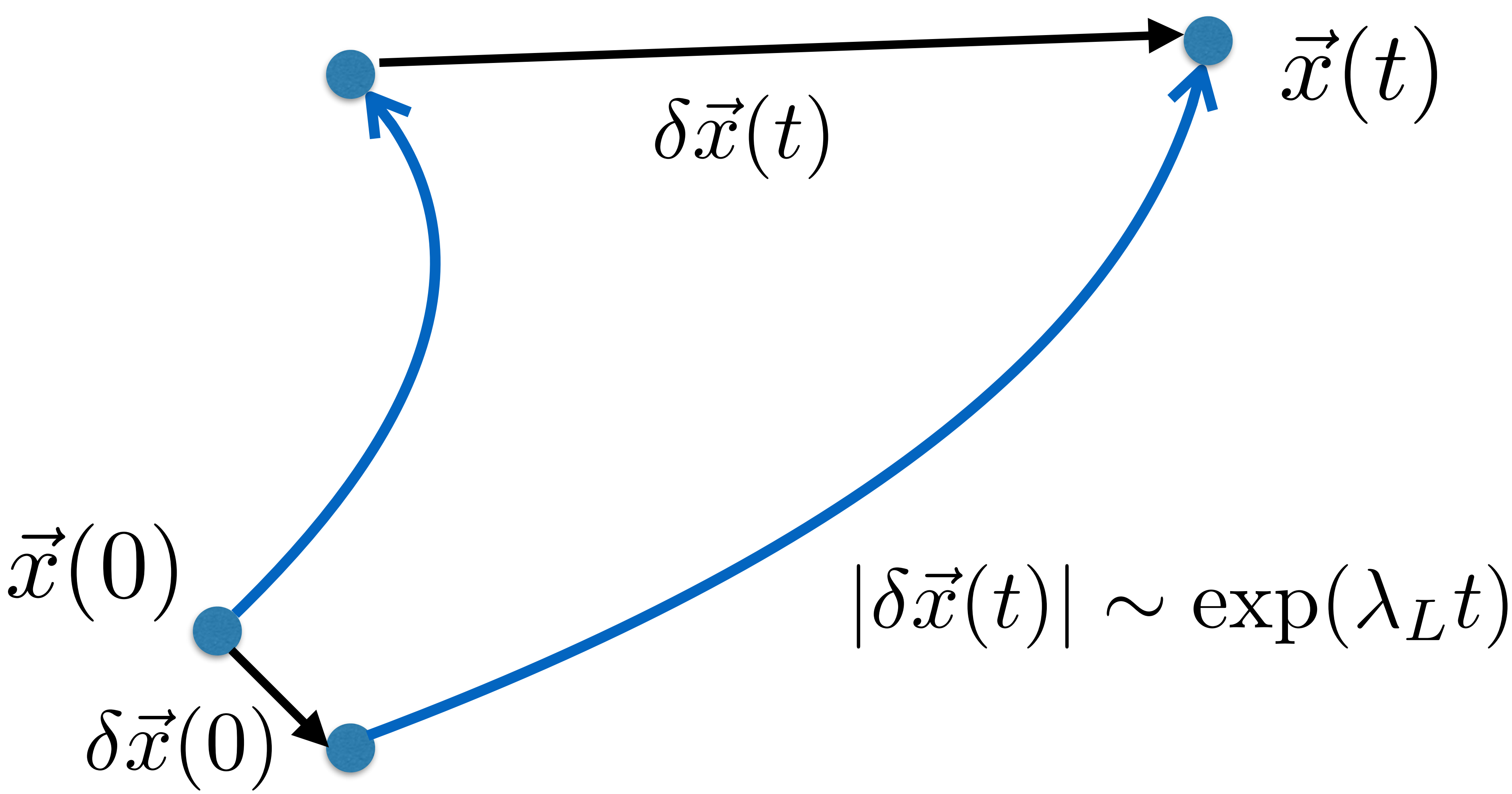
Characterization of *classical* chaos

- Sensitivity to a small perturbation.
  Lyapunov exponent $\lambda_L>0$.



Characterization of *quantum* chaos

Early time

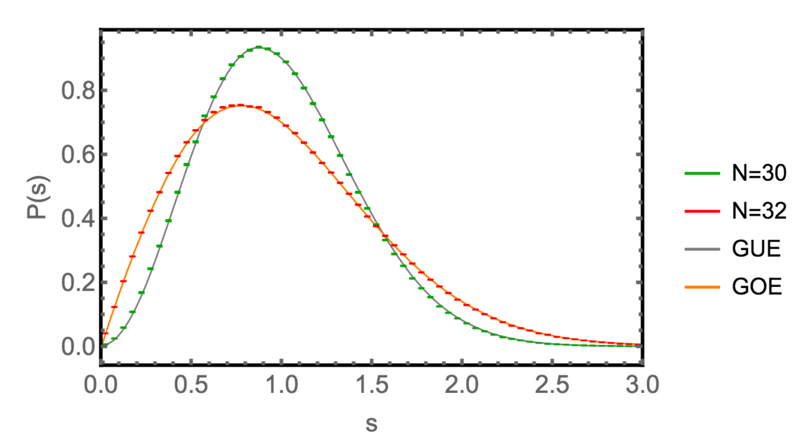
- Sensitivity to a small perturbation.
  Lyapunov exponent $\lambda_L > 0$.
  (Out-of-time-order correlation functions)

Late time

- 'Universal' energy spectrum.
  Fine-grained energy spectrum should agree with Random Matrix Theory (RMT).

'Black Holes and Random Matrices' @ STRINGS 2017 by S. Shenker

D. Stanford's talk in this conference



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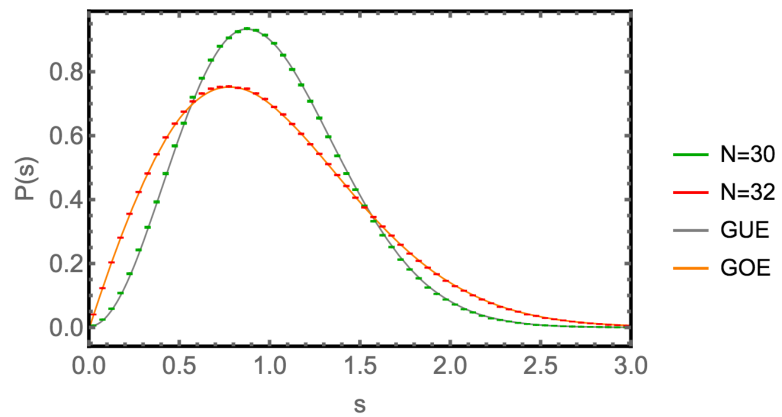

Characterization of *quantum* chaos

Also in classical chaos

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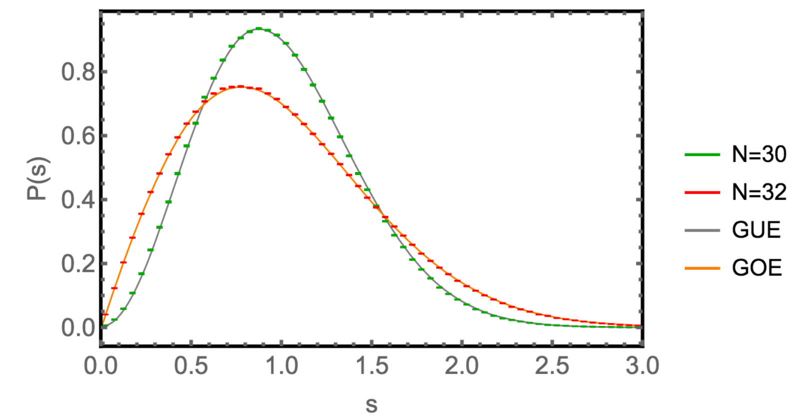

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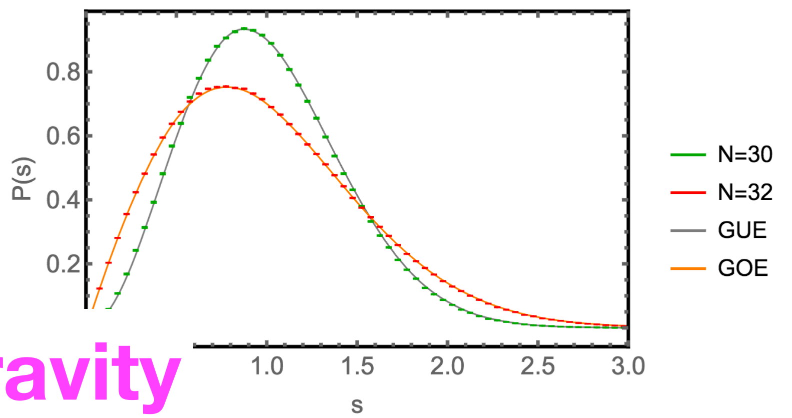


Interesting connection to quantum gravity

Lyapunov exponents

(Lyapunov spectrum)

Lyapunov Spectrum in Classical Chaos

- Classical phase space is multi-dimensional.
- Perturbation can grow or shrink to various directions.

$$z = (x, p)$$

$$M_{ij}(t) = \frac{\delta z_i(t)}{\delta z_j(0)}$$ singular value $s_i(t)$

$$L_{ij}(t) = \left[M^\dagger(t) M(t)\right]_{ij} = M^*_{ki}(t) M_{kj}(t)$$ eigenvalue $s_i(t)^2$

finite-time Lyapunov exponents $$\lambda_i(t) = \frac{1}{t} \log s_i(t)$$

Largest Exponent is not enough

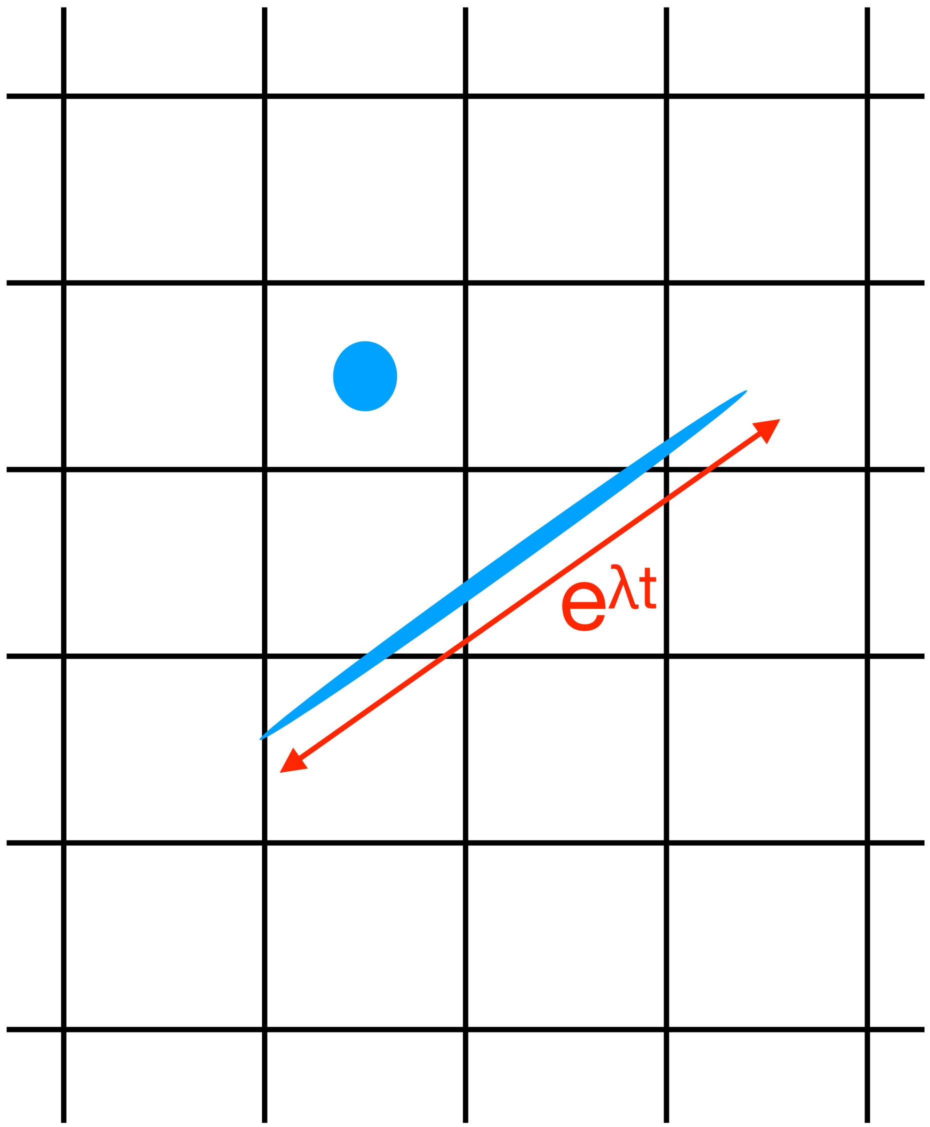
$\lambda_1=100$

$\lambda_2=\lambda_3=\ldots\lambda_{1000}=0$

$\lambda_1=\lambda_2=\ldots\lambda_{1000}=1$

Which is more chaotic?

Coarse-grained entropy and Kolmogorov-Sinai Entropy



# of cells to cover the region $\sim \prod_{\lambda>0} \exp(\lambda t)$

Coarse-grained entropy

$= \log[\text{\# of cells to cover the region}]$

$\sim$ (sum of positive λ) $\times$ t

KS entropy $=$ (sum of positive λ)

$=$ entropy production rate

Largest Exponent is not enough

$\lambda_1=100$

$\lambda_2=\lambda_3=\ldots\lambda_{1000}=0$

$\lambda_1=\lambda_2=\ldots\lambda_{1000}=1$

Which is more chaotic?

$\lambda_{1+}+\lambda_2+\ldots+\lambda_{1000}=100$

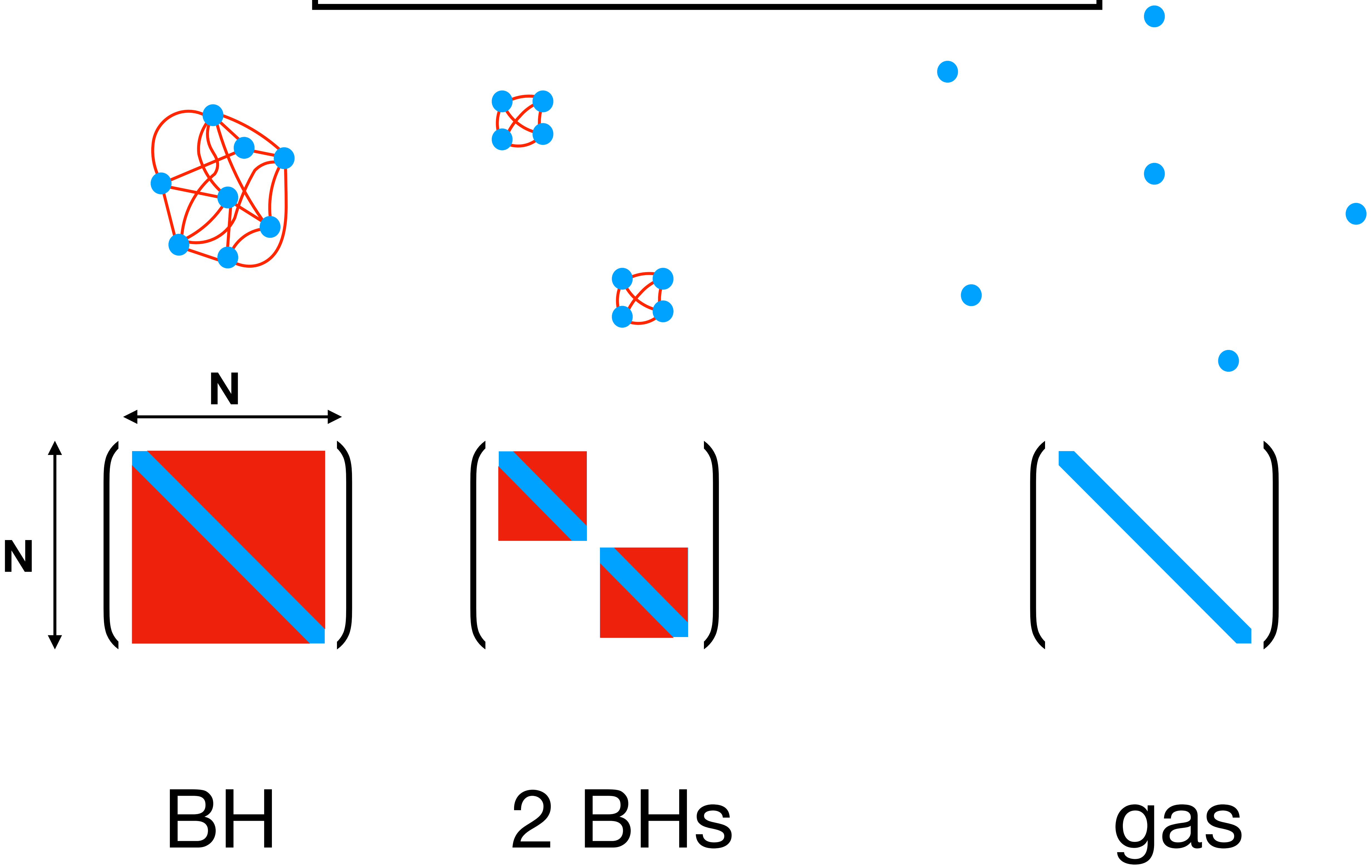
$\lambda_{1+}+\lambda_2+\ldots+\lambda_{1000}=1000$

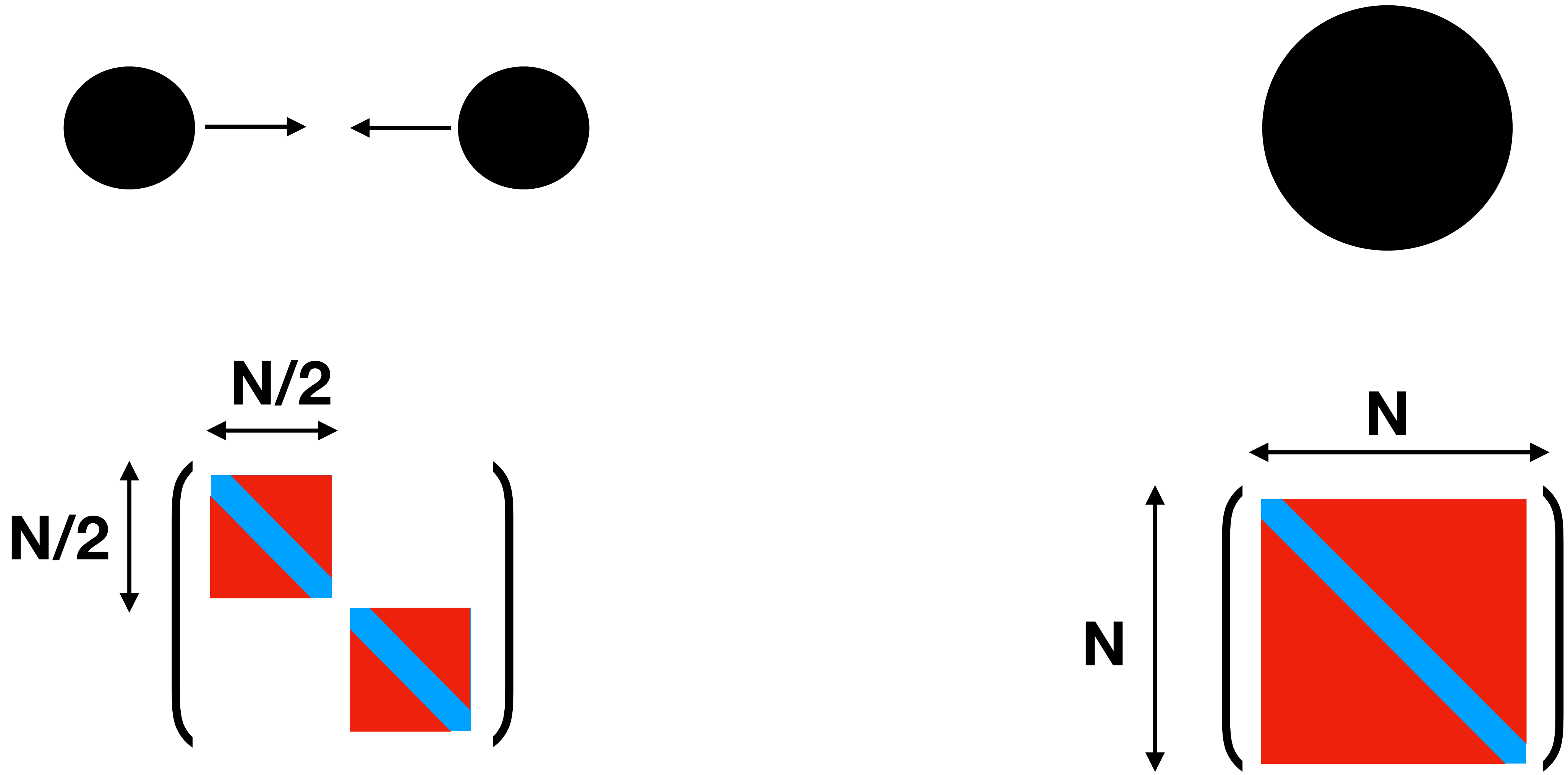
More chaotic

Bigger black hole is colder.

Bigger black hole is less chaotic?

D0-brane matrix model



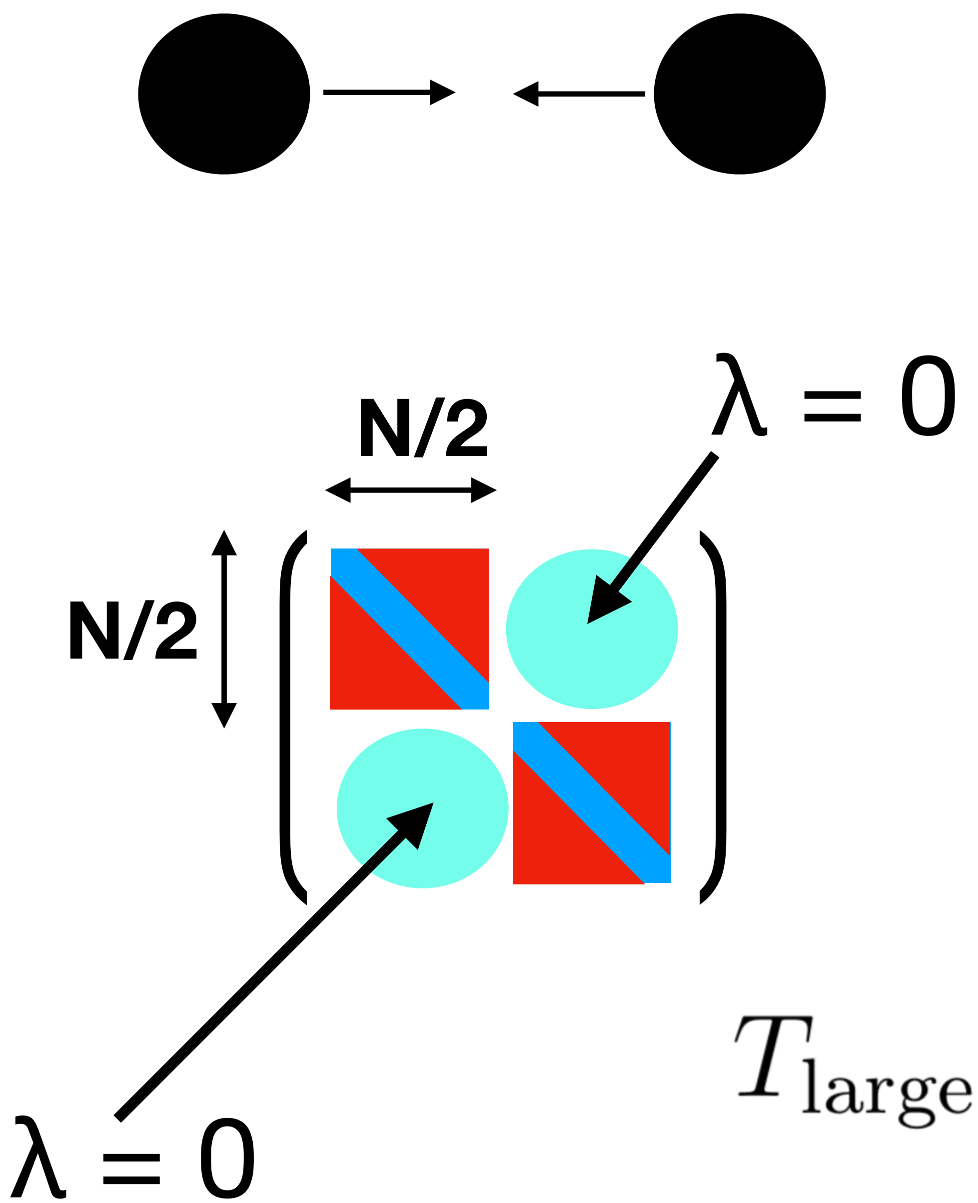


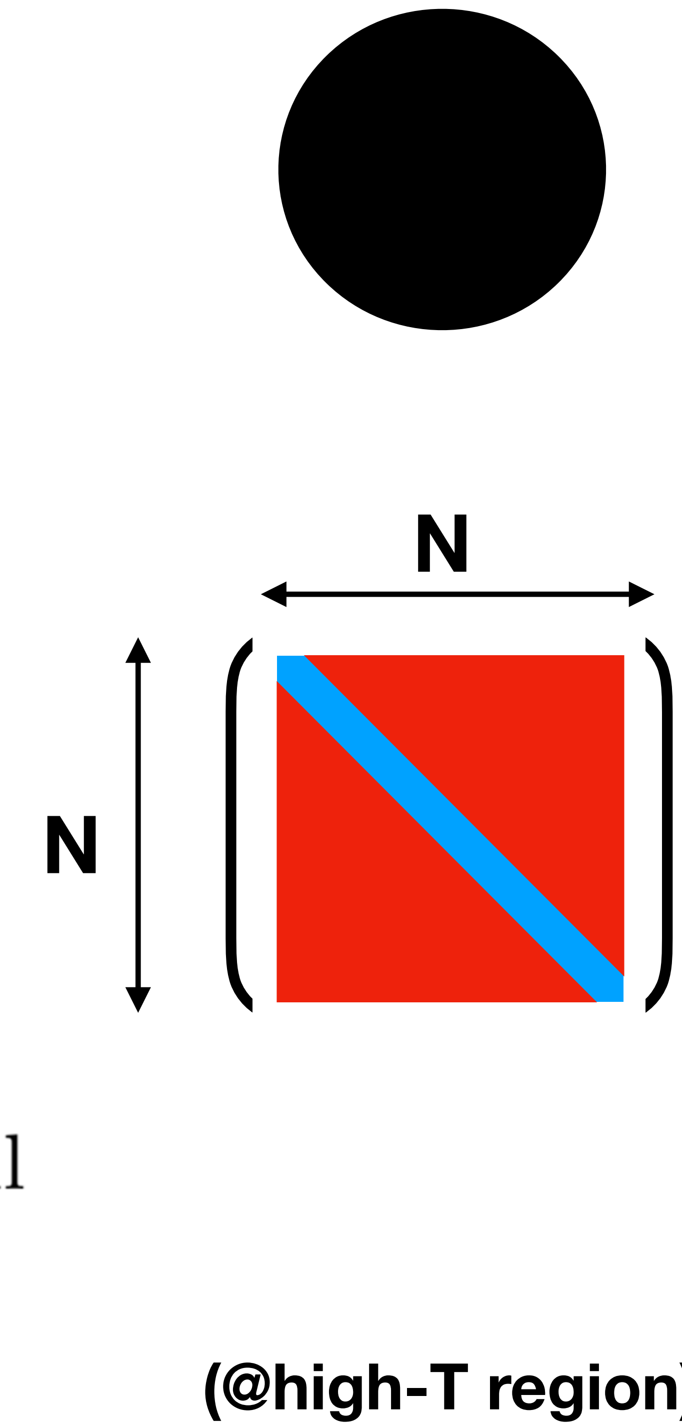
$$E \sim N^2 T_{\text{large}} = \left(\frac{N}{2}\right)^2 T_{\text{small}} \times 2$$

$$T_{\text{large}} = \frac{1}{2} T_{\text{small}}$$

$$\lambda \sim T^{1/4}$$

(@high-T region)





$$T_{\text{large}} = \frac{1}{2} T_{\text{small}}$$

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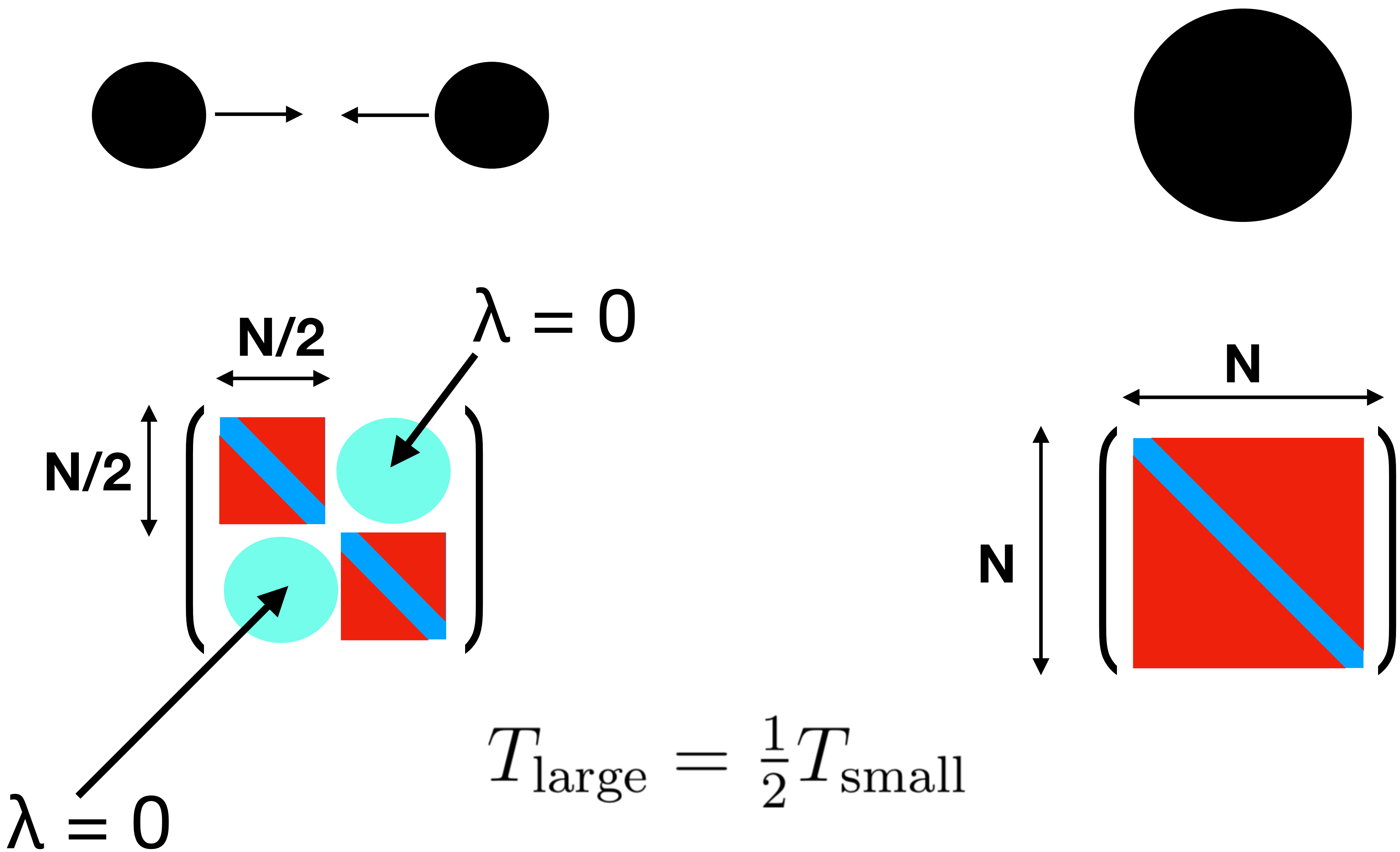


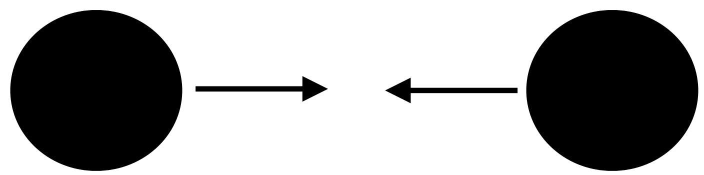
$$T_{\text{large}} = \frac{1}{2} T_{\text{small}}$$

$$\lambda \sim T^{1/4}$$

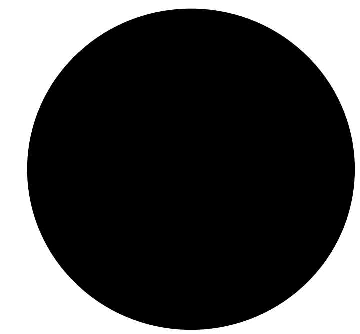
(@high-T region)

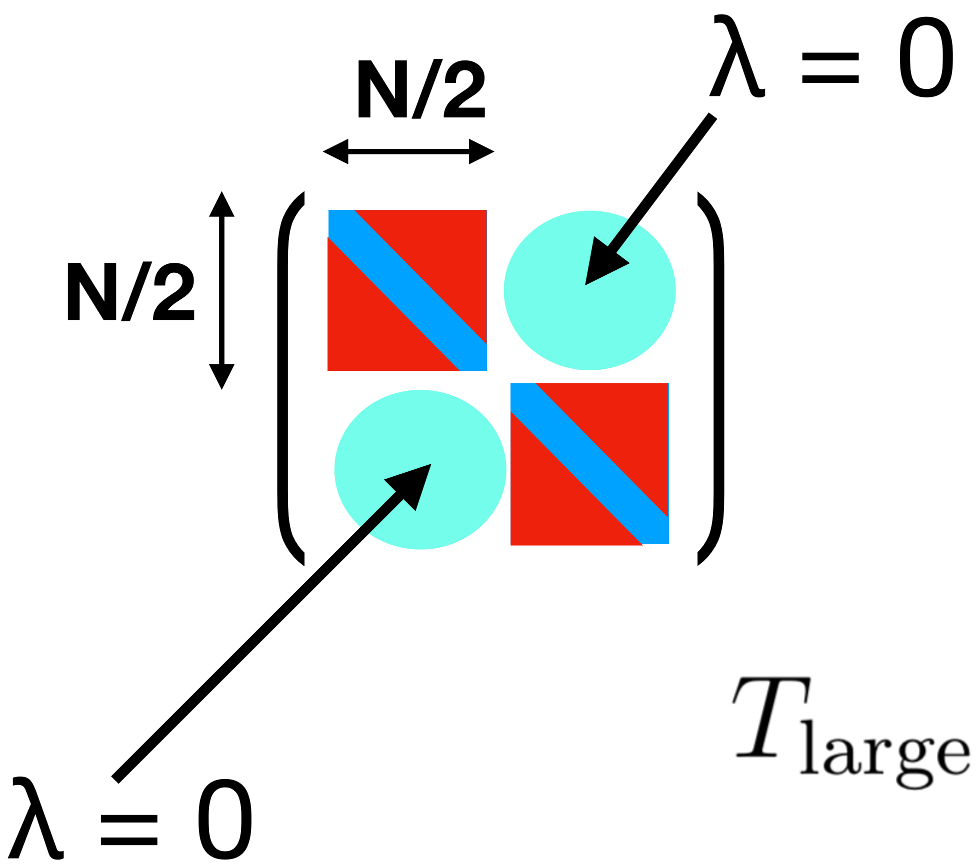
$$\boxed{2 \times \left(\frac{N}{2}\right)^2 \times T_{\text{small}}^{1/4} < N^2 \times T_{\text{large}}^{1/4}}$$

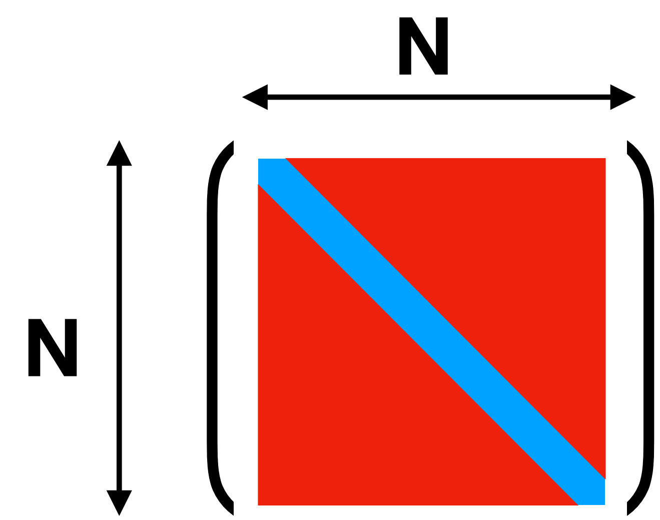
Similar calculation is doable at low-T and also for other theories (Berkowitz-MH-Maltz 2016)



More chaotic







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Plan

- Universality of *classical* Lyapunov spectrum

MH, Shimada, Tezuka, PRE 2018

- Universality of *quantum* Lyapunov spectrum

Gharibyan, MH, Swingle, Tezuka, in progress

Lyapunov Spectrum

$$z = (x, p)$$

$$M_{ij}(t) = \frac{\delta z_i(t)}{\delta z_j(0)}$$ → singular value $s_i(t)$

$$L_{ij}(t) = \left[M^\dagger(t) M(t)\right]_{ij} = M^*_{ki}(t) M_{kj}(t)$$

→ eigenvalue $s_i(t)^2$

finite-time Lyapunov exponents $$\lambda_i(t) = \frac{1}{t} \log s_i(t)$$

Lyapunov Spectrum

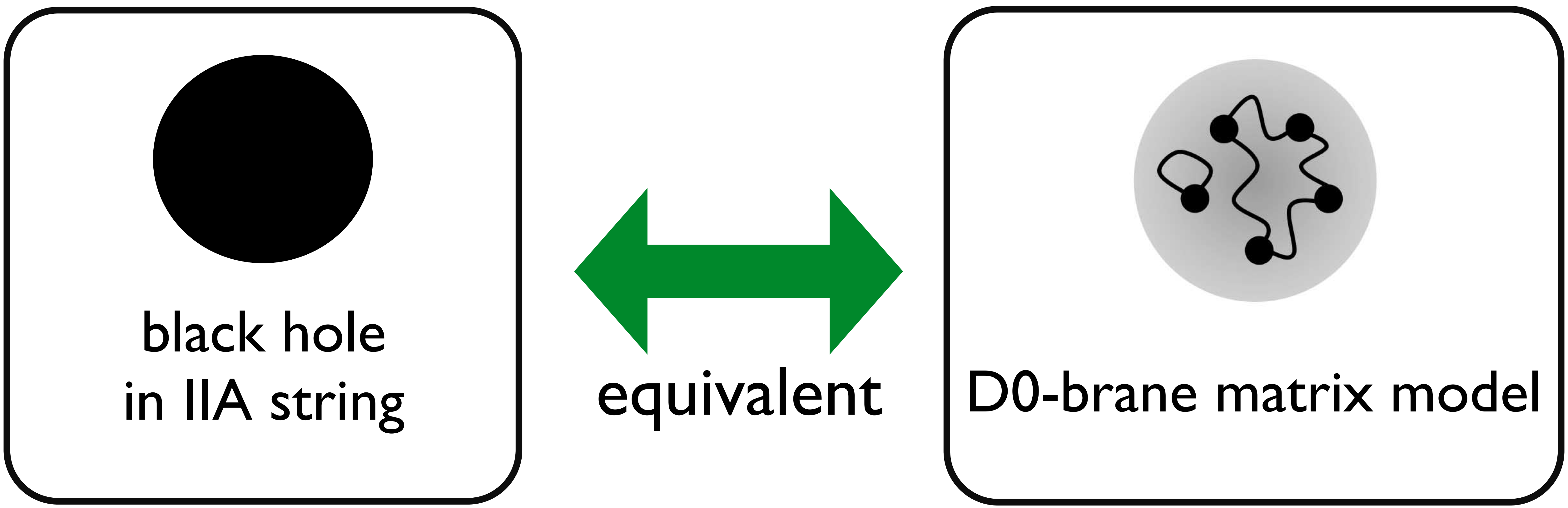
$$
\begin{aligned}
M_{ij}(t) &= \frac{\delta z_i(t)}{\delta z_j(0)} \\
&= \frac{\delta z_i(t)}{\delta z_k(t-\Delta t)} \cdot \frac{\delta z_k(t-\Delta t)}{\delta z_l(t-2\Delta t)} \cdots \frac{\delta z_m(\Delta t)}{\delta z_j(0)}
\end{aligned}
$$

Easily to calculate with good precision

D0-brane matrix model

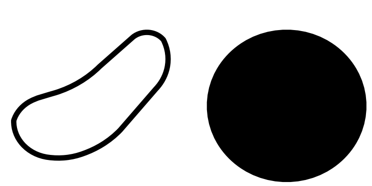
negligible at high-T

$$L = \frac{1}{2g_{YM}^2}\mathrm{Tr}\left\{(D_t X_M)^2 + [X_M, X_{M'}]^2 + \cancel{i\bar{\psi}^\alpha D_t \psi^\beta + \bar{\psi}^\alpha \gamma^M_{\alpha\beta}[X_M, \psi^\beta]}\right\}$$

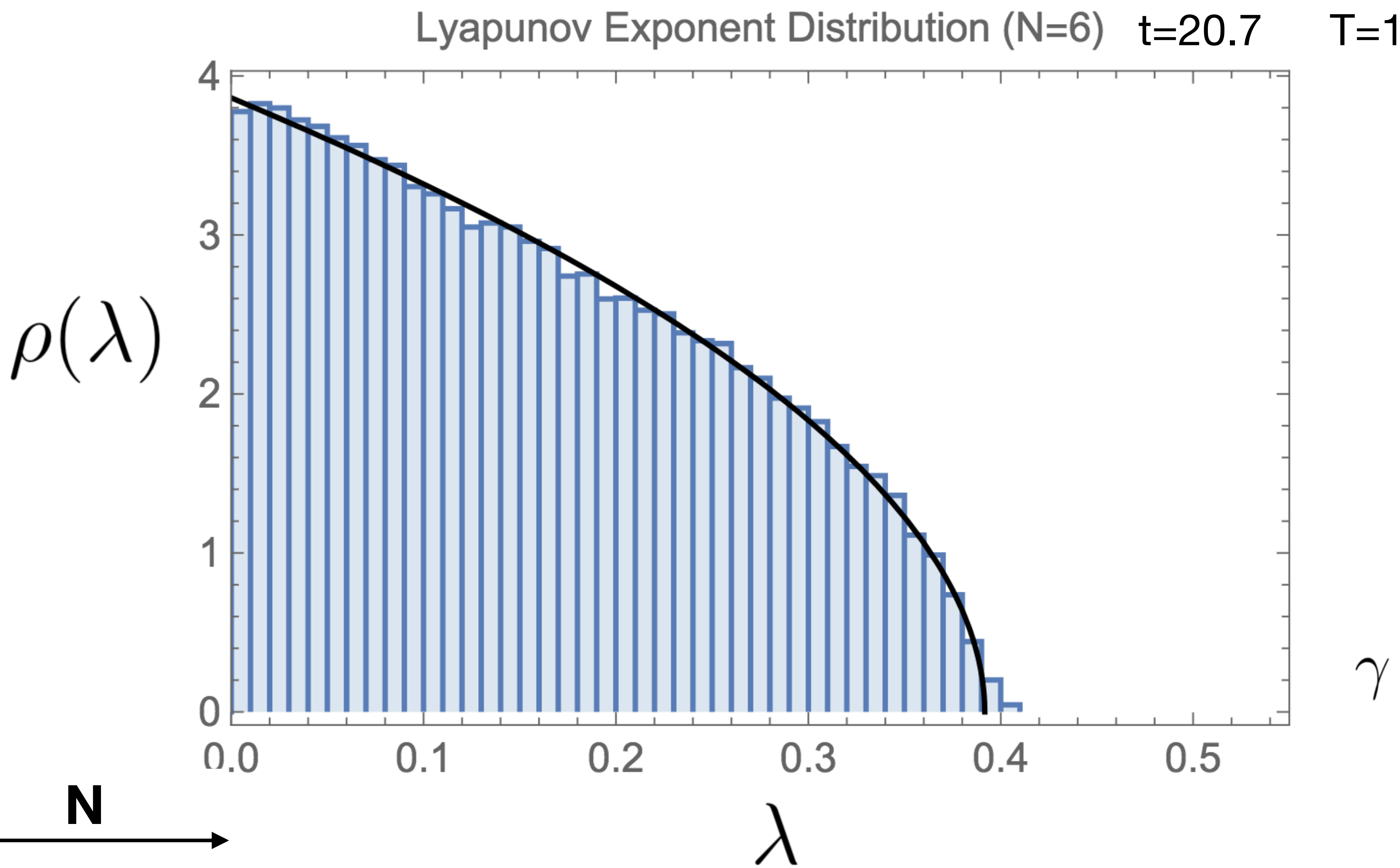


high-T = ‘stringy’

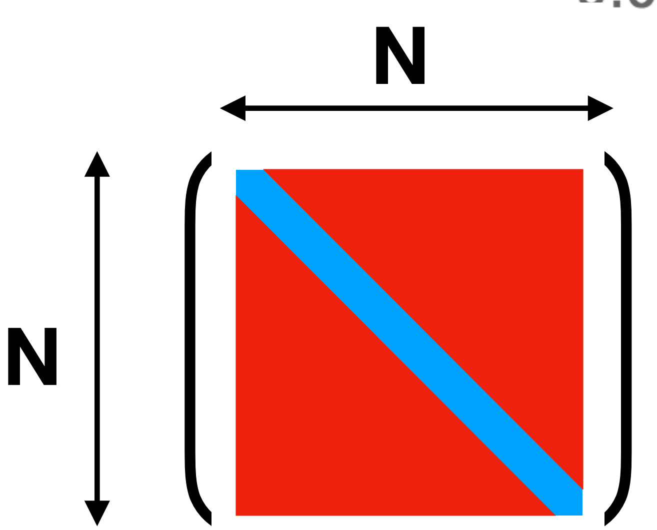
high-T = classical

string  BH

Gur Ari-MH-Shenker, JHEP2016



$\gamma = 0.513$



Fitting ansatz $\rho(\lambda) = \dfrac{(\gamma+1)(\lambda_{max}-\lambda)^{\gamma}}{\lambda_{max}^{\gamma+1}}$

RMT vs Classical Chaos

- The correlation of the finite-time Lyapunov exponents may have a universal behavior?

(Some hints found in the previous study by Gur-Ari, MH, Shenker)

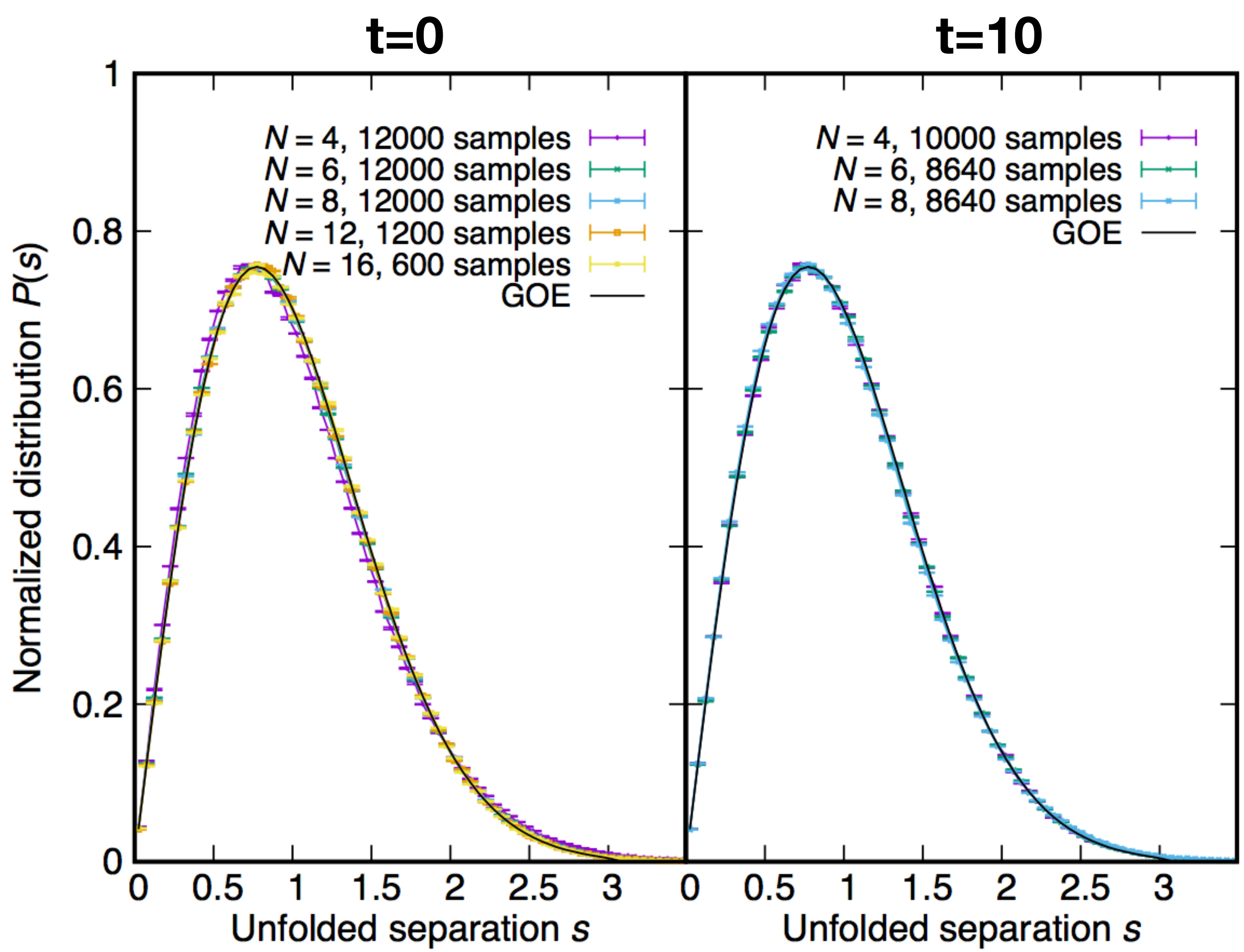
$$\lambda_1 < \lambda_2 < \cdots < \lambda_N$$
$$s_i = \lambda_{i+1} - \lambda_i$$

(different from $s_i = \exp(\lambda_i t)$, sorry for using the same letter!)

- $N \to \infty$ before $t \to \infty$

(In chaos community, often $t \to \infty$ is taken first.)

GOE-distribution at any time

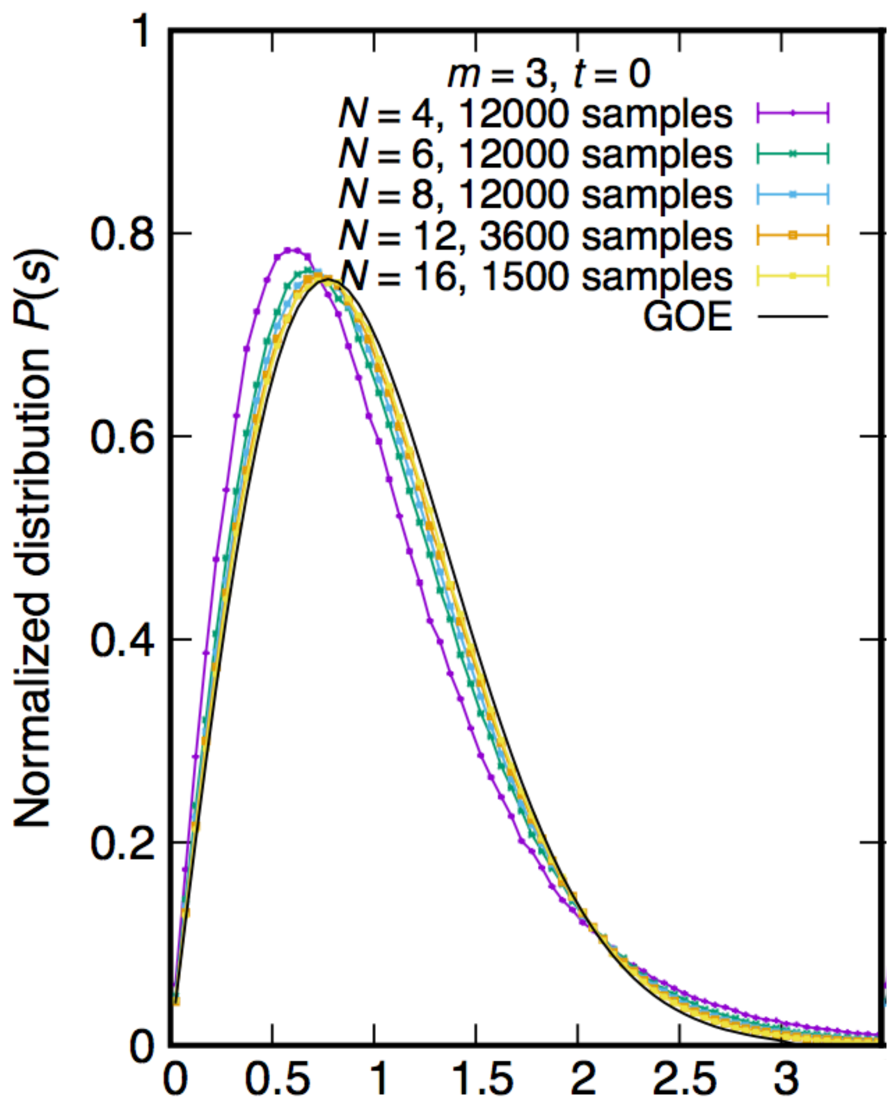


$$s_i = \lambda_{i+1} - \lambda_i$$

Lyapunov exponents are described by RMT

M.H.-Shimada-Tezuka, PRE 2018

with a mass term (→no gravity interpretation), GOE is gone, at t=0.



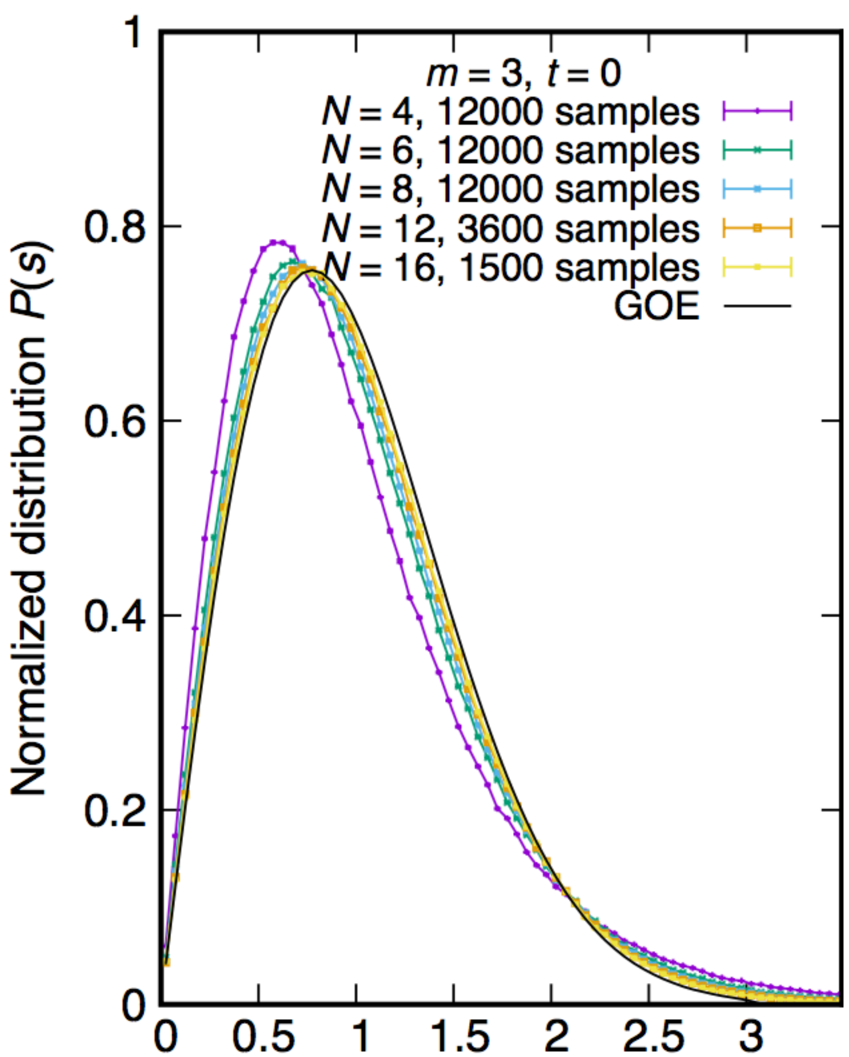
$$s_i = \lambda_{i+1} - \lambda_i$$

m=3, t=0

$$\Delta L = -\frac{Nm^2}{2}\mathrm{Tr}X_M^2$$

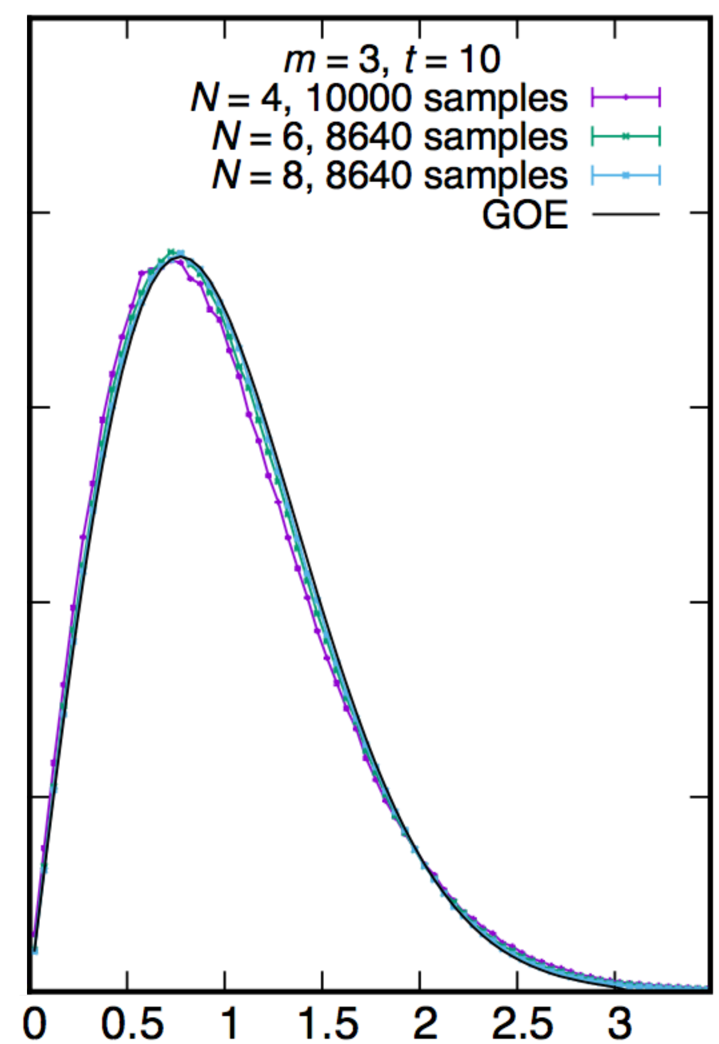
But GOE is back at later time

t=0



$$s_i = \lambda_{i+1} - \lambda_i$$

t=3



$$s_i = \lambda_{i+1} - \lambda_i$$

$$\Delta L = -\frac{Nm^2}{2}\mathrm{Tr}X_M^2$$

Summary of numerical observations

- **Universality beyond nearest-neighbor can be checked. (Spectral Form Factor)**
- **D0-brane matrix model — RMT already t=0**

  Maybe a special property of quantum gravitational systems?

- **Other systems — not RMT at t=0, but gradually converges to RMT.**

  Likely to be a universal property in classical chaos.
  Generalization to quantum theory?

- **So far we have looked at only the bulk of the spectrum; not the edge.**

Early-time universality in quantum chaos

Gharibyan, MH, Swingle, Tezuka, in progress

- There is no consensus for the definition of ‘quantum Lyapunov spectrum’
- Let’s try the simplest choice:

$$M_{ij}(t) = \frac{\delta z_i(t)}{\delta z_j(0)}$$

$$\hat{M}_{ij} = \sqrt{-1}\left[\hat{z}_i(t), \hat{\Pi}_j(0)\right]$$

$$L_{ij}(t) = M^*_{ki}(t) M_{kj}(t)$$

$$L^{(\phi)}_{ij}(t) = \langle\phi|\hat{M}^*_{ki}(t)\hat{M}_{kj}(t)|\phi\rangle$$

$$\lambda_i(t) = \frac{1}{t}\log s_i(t)$$

$\hat{M}_{ij}(t)|\phi\rangle$ grows exponentially

$\langle\phi|\hat{M}_{ij}(t)|\phi\rangle$ cannot capture the growth

SYK model

$$\hat{H} = \sqrt{\frac{6}{N^3}} \sum_{i<j<k<l} J_{ijkl} \hat{\psi}_i \hat{\psi}_j \hat{\psi}_k \hat{\psi}_l + \frac{\sqrt{-1}}{\sqrt{N}} \sum_{i<j} K_{ij} \hat{\psi}_i \hat{\psi}_j$$

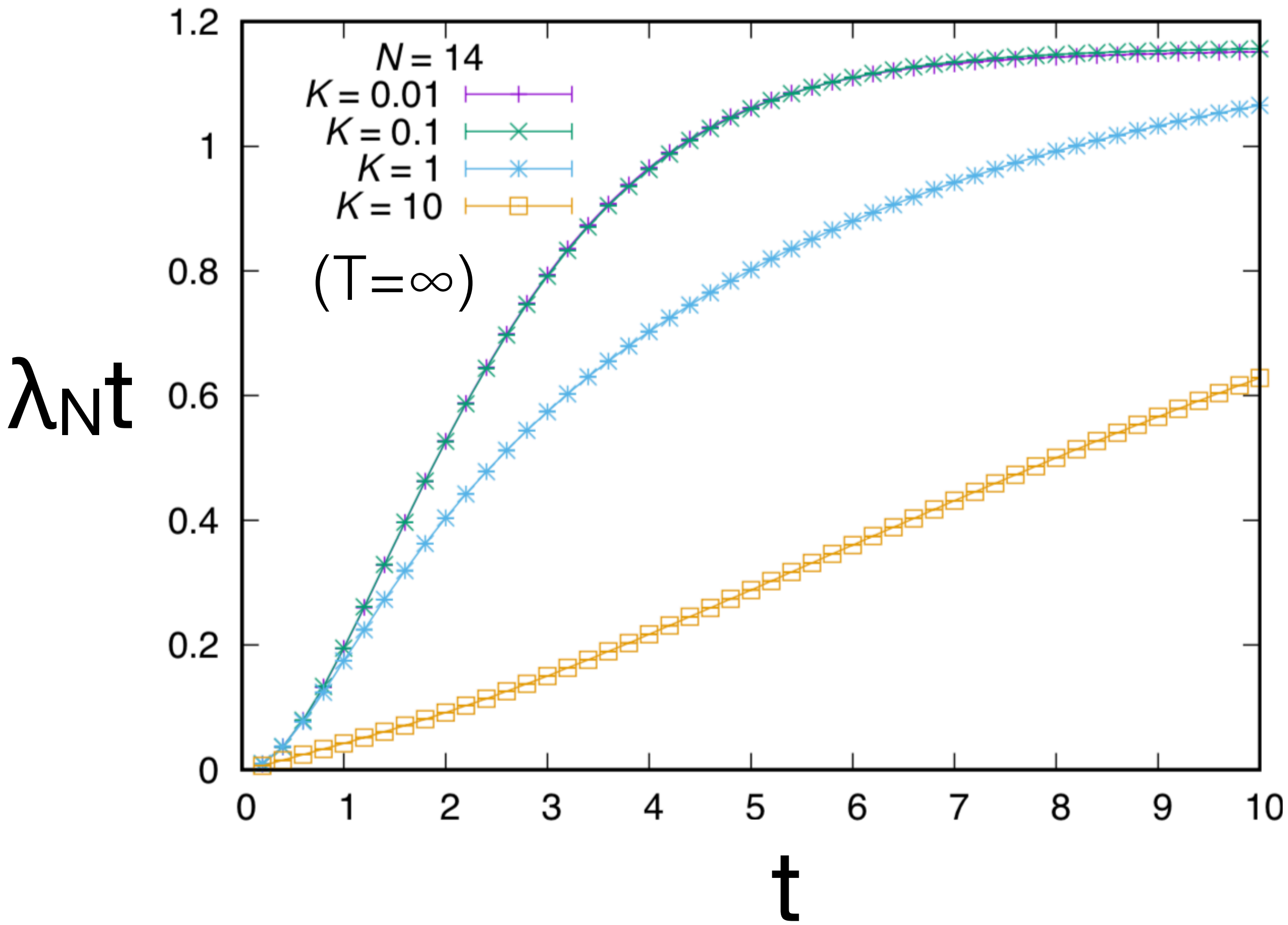
maximally chaotic

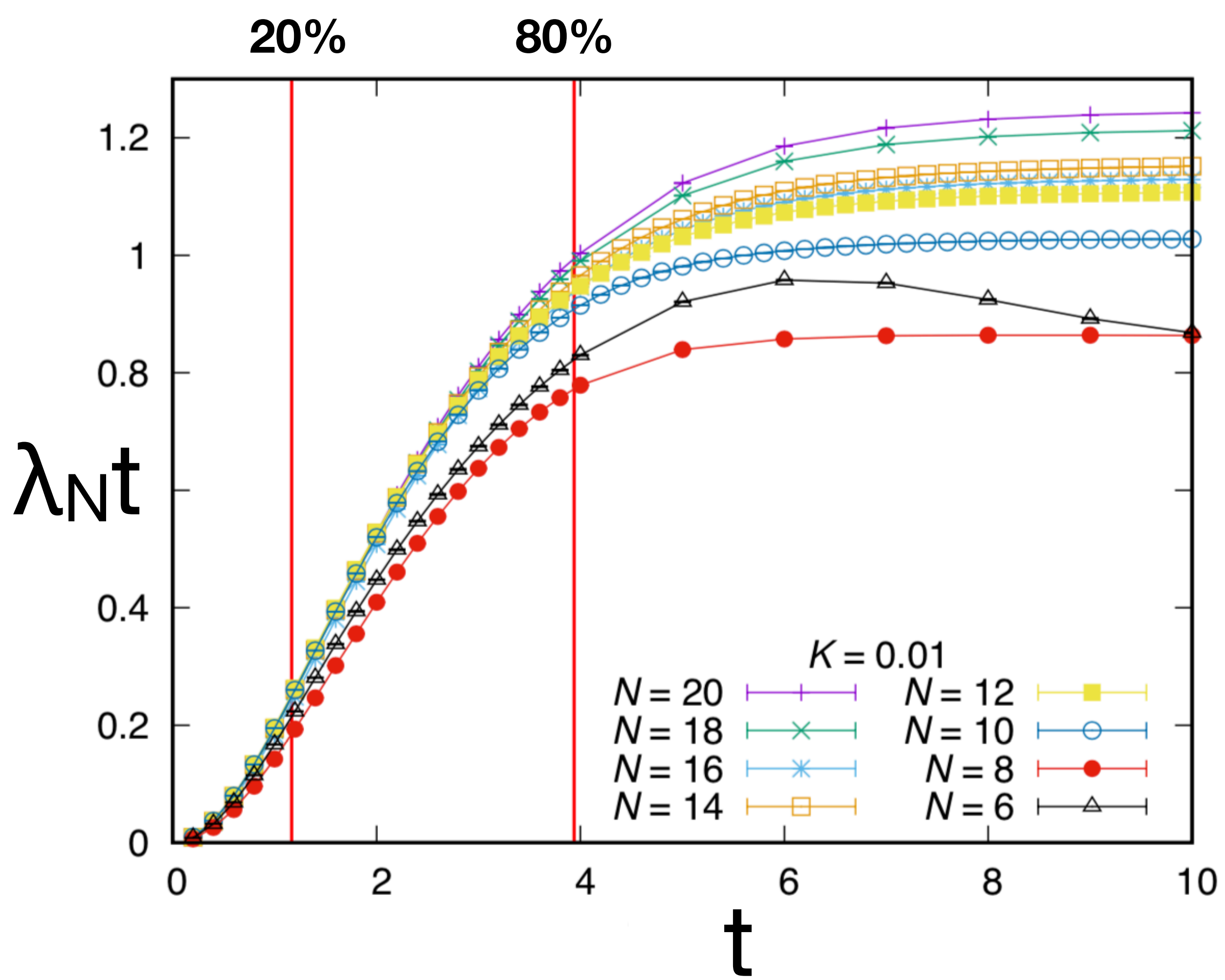
integrable

$$\hat{M}_{ij}(t) = \{\hat{\psi}_i(t), \hat{\psi}_j(0)\}$$

$$e^{2\lambda^{(\mathrm{OTOC})} t} = \frac{1}{N} \sum_{i,j} \langle \phi | \{\hat{\psi}_i(t), \hat{\psi}_j(0)\}^2 | \phi \rangle = \frac{1}{N} \sum_i e^{2\lambda_i t}$$

Lyapunov growth



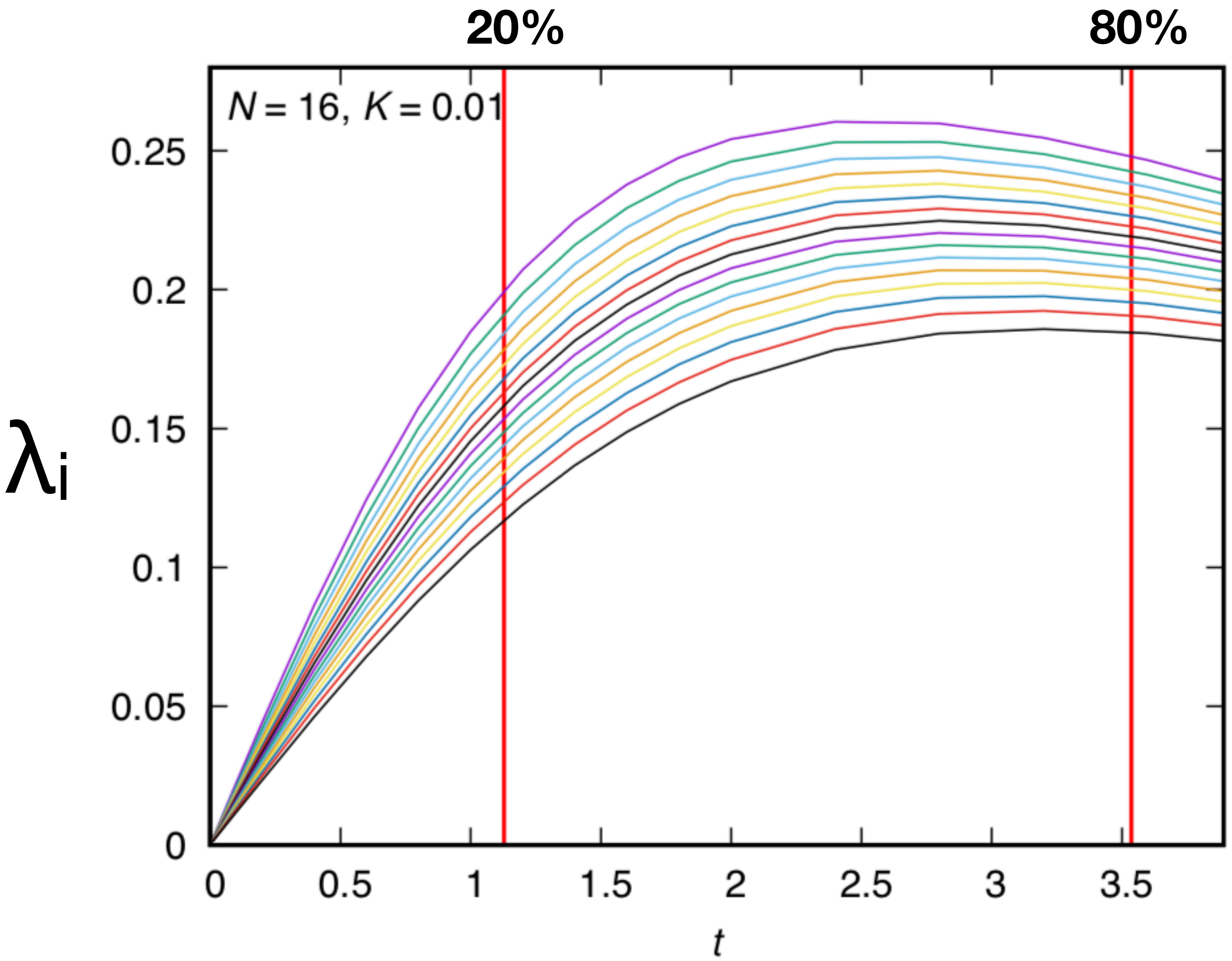


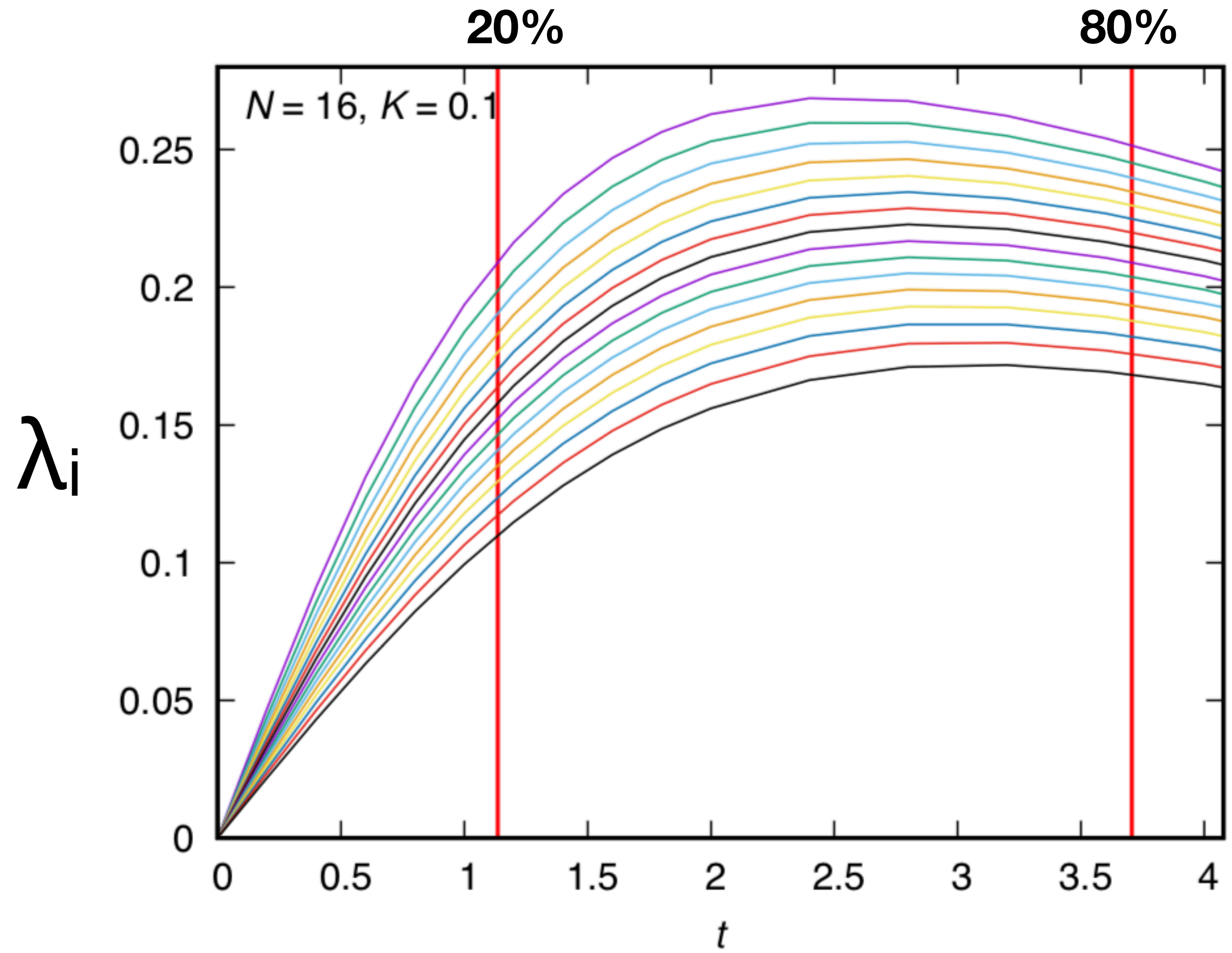
$$\lambda_1 < \lambda_2 < \cdots < \lambda_N$$

$$\lambda_i t = \log s_i(t)$$

Preliminary

Lyapunov growth





Preliminary

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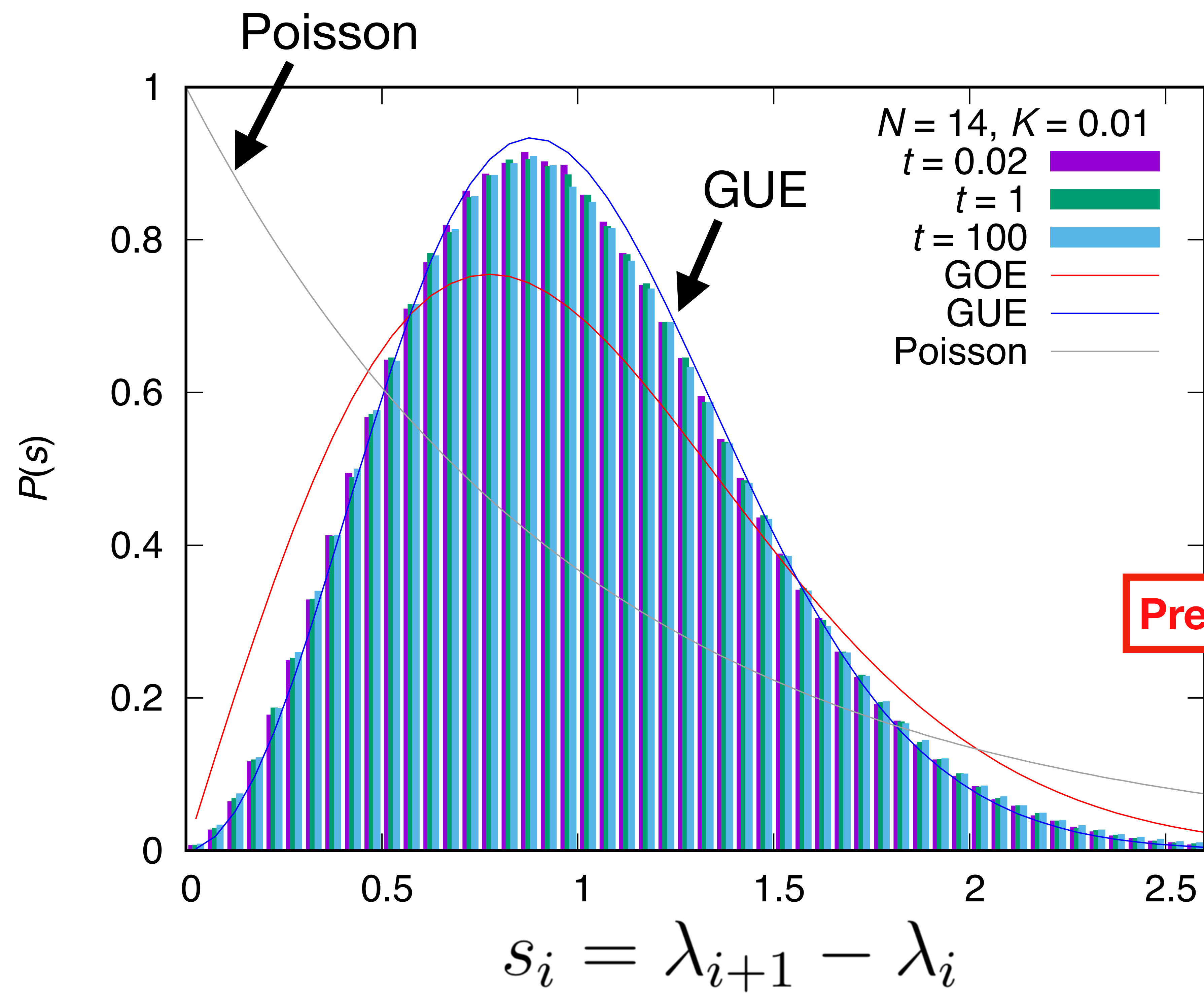
RMT behavior



Preliminary

RMT behavior

- $K > 0 \to$ chaotic at high energy, non-chaotic at low energy

(Garcia-Garcia, Loureiro, Romero-Bermudez, Tezuka, 2017)

- Our numerical data suggests:

Chaotic states → RMT

non-chaotic states → Poisson

- Brownian circuit version is consistent with this interpretation.

Spin chain (XXZ model)

$$\hat{H} = \sum_{i=1}^{N_{site}} \left(\frac{1}{4} \vec{\sigma}_i \vec{\sigma}_{i+1} + \frac{\omega_i}{2} \sigma_{z,i} \right)$$

XXX model

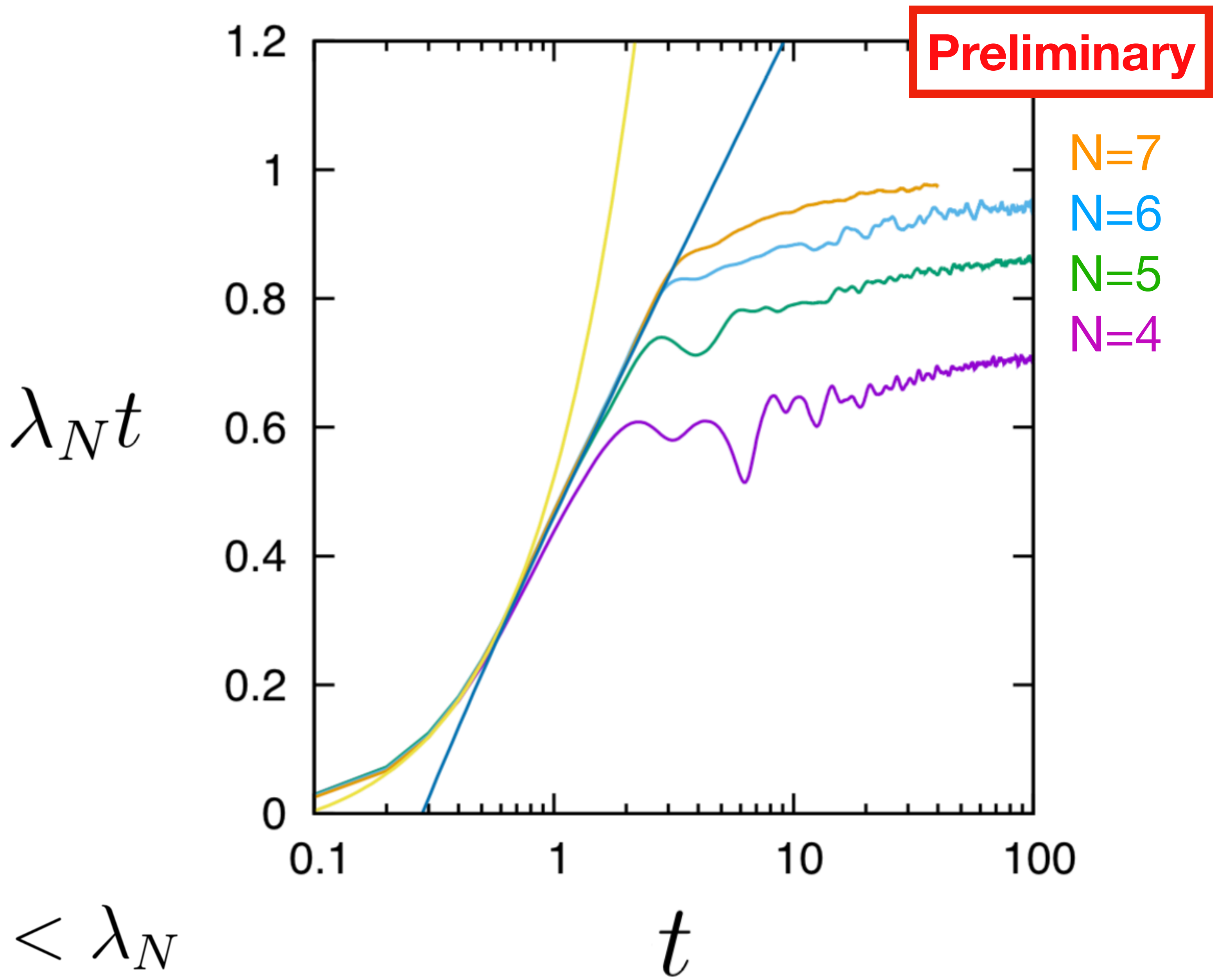
random magnetic field

$$-w \leq w_i \leq +w$$

- Ergodic at small w
- Many-body localized (MBL) at large w

$$\hat{M}_{ij} \equiv [\sigma_{+,i}(t), \sigma_{-,j}(0)]$$

Spin chain (XXZ model)

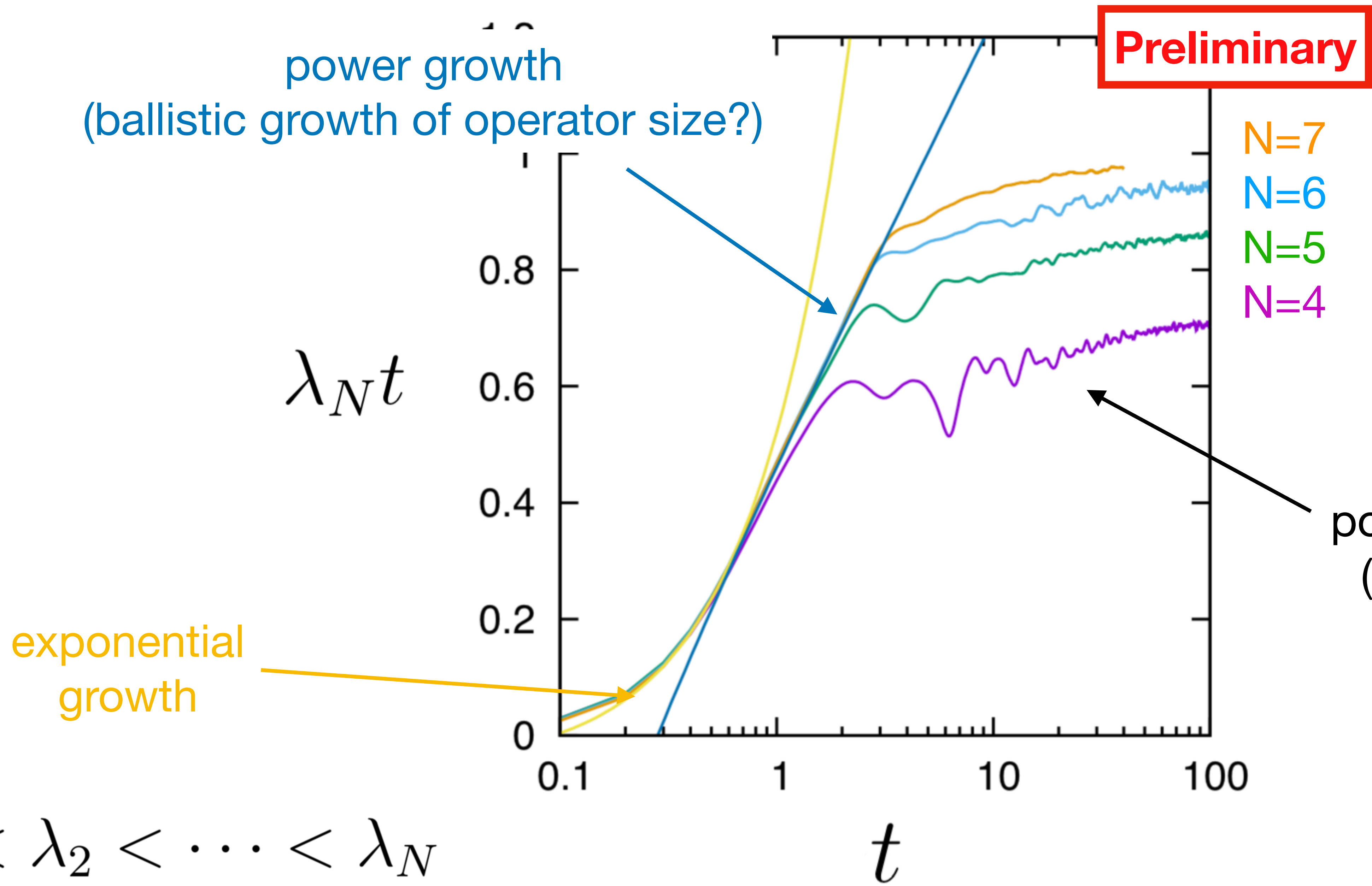


$$\lambda_1 < \lambda_2 < \cdots < \lambda_N$$

$$\lambda_i t = \log s_i(t)$$

w=0.5 (ergodic phase), T=∞

Spin chain (XXZ model)

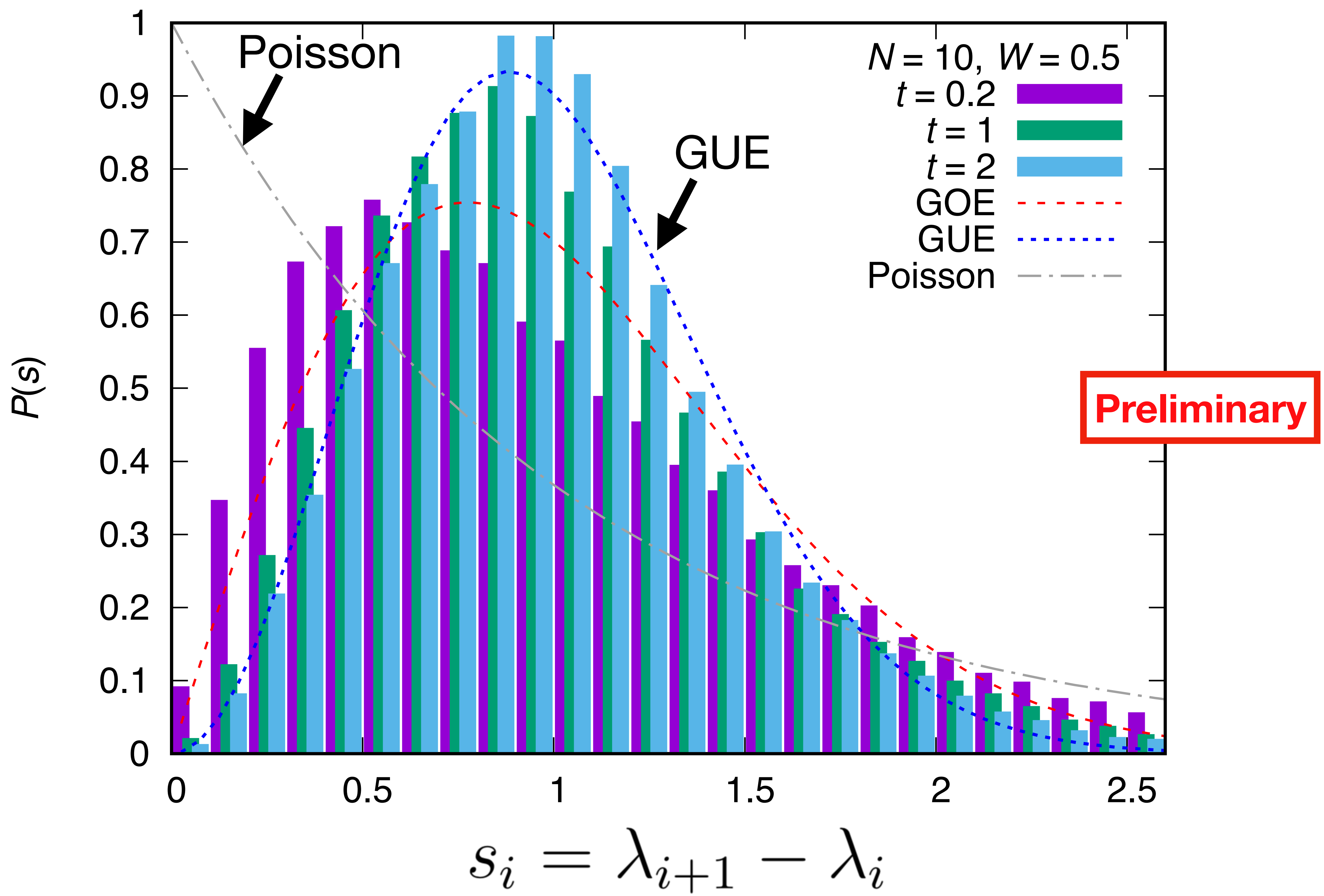


$$\lambda_1 < \lambda_2 < \cdots < \lambda_N$$

$$\lambda_i t = \log s_i(t)$$

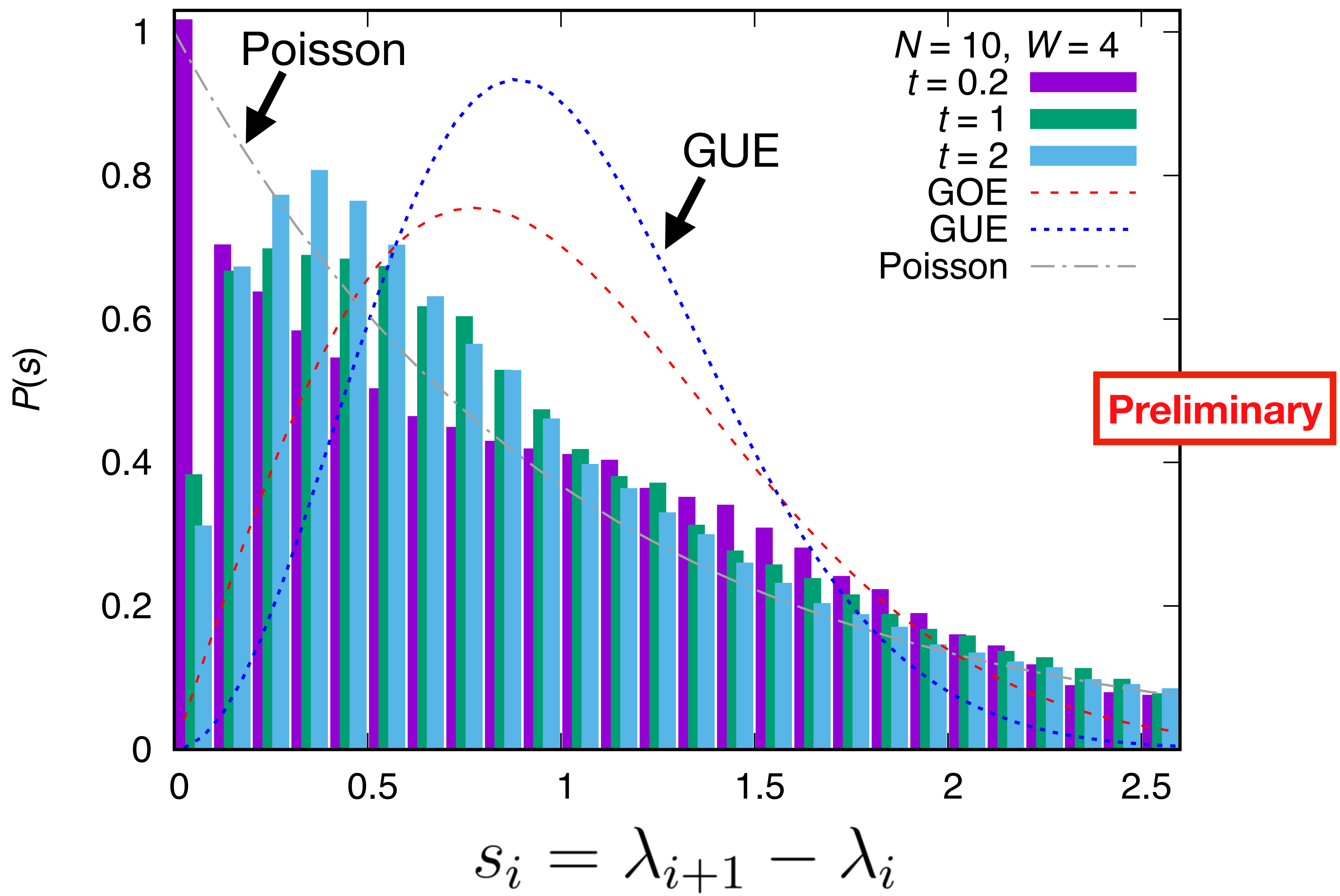
w=0.5 (ergodic phase), T=∞

RMT vs Lyapunov spectrum in XXZ model



N=10, w=0.5 (ergodic phase)

RMT vs Lyapunov spectrum in XXZ model



N=10, w=4.0 (MBL phase)

Summary of numerical observations

- Classical chaos

  D0 matrix model — ‘strongly’ universal
  Other chaotic systems — universal

- Quantum chaos

  SYK — ‘strongly’ universal
  Other chaotic systems — universal
  MBL — not universal (Poisson-like)

- Lyapunov growth can be seen precisely.

Conclusion & Outlook

- The largest Lyapunov exponent is not enough.
- Lyapunov spectrum captures physics more precisely.
- New universality.
- Black hole is (probably) special.
- What is the mechanism?
- How can we formulate the spectrum in gravity side?
- Relation to the late time universality (energy spectrum)?
- ‘KS entropy’ vs EE growth rate?
- Generalization of the chaos bound to KS entropy?