Calabi-Yau fibrations, finiteness, and string dualities

Lara B. Anderson

Virginia Tech

Work done in collaboration with:

(L.A, J. Gray, X. Gao, B. Hammack, S. J. Lee)

- arXiv:1805.05497,1708.07907,1608.07555,1608.07554

STRINGS 2018

OIST, Okinawa

June 29th, 2018

Motivation

Goal: A better understanding of possible string effective theories

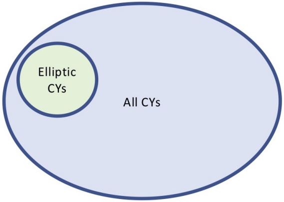
- Within string compactifications, *which EFTs?* $\leftrightarrow$ *which geometries?*
- Even the simplest class of compactification manifolds $\rightarrow$ Calabi-Yau (complex) n-folds, unclassified for $n > 2$. Finite?
- Even weaker question: is the topology (i.e. for CY 3-folds values of $h^{1,1}, h^{2,1}, Ch_2(TX)$, etc) of Calabi-Yau n-folds bounded?
- In this talk: I'll outline one approach these questions via CY fibrations.
- Quick reminder:
  - A fibration is a surjective morphism $\pi : X \rightarrow B$ such that for almost all points $b, b' \in B$, $\pi^{-1}(b) \simeq \pi^{-1}(b')$ (homotopy lifting)
  - A continuous map $\sigma : B \rightarrow X$ s.t. $\pi(\sigma(b)) = b$ for all $b \in B$ is called a *section* to the fibration.

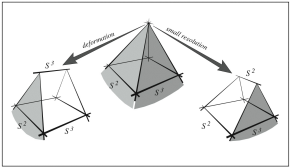
Why CY fibrations?

- Fibrations play a key role in string dualities: F-theory/M-theory ($\mathbb{E}$-fiber), Heterotic/F-theory ($\mathbb{E}$/K3-Fiber), Heterotic/Type IIA, etc.
- Fiber/base decomposition simplifies geometry, increases calculability (form of intersection numbers, metric, etc).
- Theorem (Gross, Grassi, '93): The set of genus one fibered CY 3-folds is finite.
- (Di Cerbo + Svaldi, 2016): Finite # of bases for elliptically fibered CY 4- and 5-folds.
- (Terminology: If a section exists, torus/genus one fibration $\Rightarrow$ "elliptic")

A few results on CY fibrations:

- CY n-folds can only be fibered by m-folds with $c_1 = 0$, with $0 < m < n$.
- The existence of a fibration is a deformation invariant statement for CY n-folds with $n > 2$: If X is a genus-1 fibered CY manifold with $H^2(X, \mathcal{O}_X) = 0$, then every small deformation of X is also genus 1 fibered.
- Motivation behind Gross' theorem: Can finiteness of genus one fibered CY threefolds be used to say anything about the finiteness of *all CY 3-folds*?



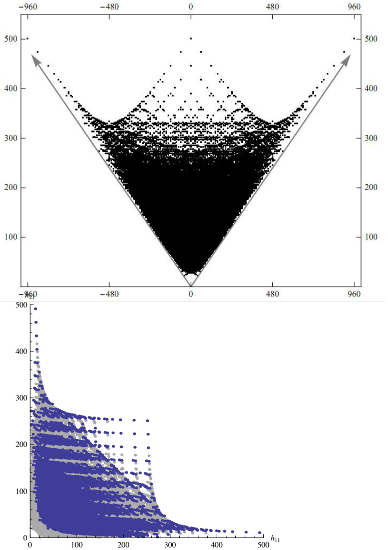


- Can geometric transitions linking elliptically fibered vs. non-fibered CYs be characterized/classified?

Observations

- Consider your favorite dataset of CYs within the order $\sim 10^9$ known (i.e. CICYs (Candelas, et al '87), toric hypersurfaces (Kreuzer/Skarke, 2000), etc)
- Immediate Observation 1: Almost all known CY 3- and 4- folds are fibered (with elliptic, K3, CY3, fibrations, etc)
- Immediate Observation 2: generic manifolds do not admit just one elliptic fibration, they admit many (order 10s, 100s, 1000s...more).
- What can we make of this? Let's take it one at a time...

Genericity of fibrations for known datasets



(from Taylor 1205.0952)

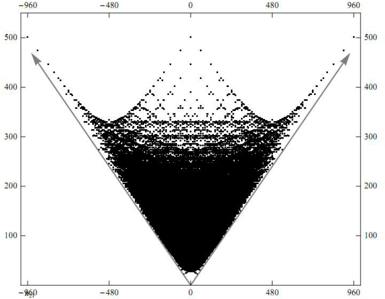
- Toric hypersurfaces: (Rohsiepe, Braun, Morrison, Taylor, Huang...)
- For the CICY 3-folds 99.3%. E.g.

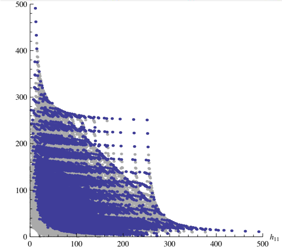
$$\left[\begin{array}{c|cc} \mathbb{P}^2 & 0 & 3 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 3 & 0 \end{array}\right]^{19,19}$$

- CICYs 4-folds in products of projective spaces 99.9% (921, 020 out of 921, 497) (Gray, Haupt, Lukas '14)

$$\left[\begin{array}{c|ccccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{P}^3 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \mathbb{P}^4 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \mathbb{P}^1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Genericity of fibrations for known datasets





(from Taylor 1205.0952)

- Toric hypersurfaces: (Rohsiepe, Braun, Morrison, Taylor, Huang...)
- For the CICY 3-folds 99.3% E.g.

$$\left[\begin{array}{c|cc} \mathbb{P}^2 & 0 & 3 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 3 & 0 \end{array}\right]^{19,19}$$

- CICYs 4-folds in products of projective spaces 99.9% $(921,020$ out of $921,497)$ (Gray, Haupt, Lukas '14)

$$\left[\begin{array}{c|ccccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{P}^3 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \mathbb{P}^4 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \mathbb{P}^1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

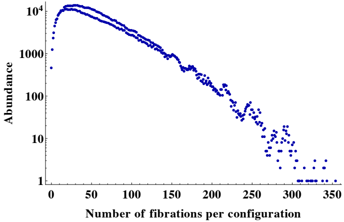
Multiplicity of fibrations

- Point of interest: When such manifolds have fibrations, they generically do not have just one...
- In fact there can be **many**
- E.g.

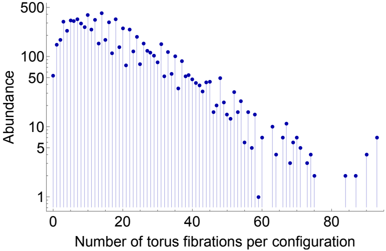
1) $\left[\begin{array}{c|ccc} \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right]$

2) $\left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \end{array}\right]$

- Thus far: "Obvious" fibrations



CICY 4-folds (Gray, et al 1405.2073)($\sim 10^8$ fibrations)



CICY 3-folds ($\sim$ 140, 000 fibrations)

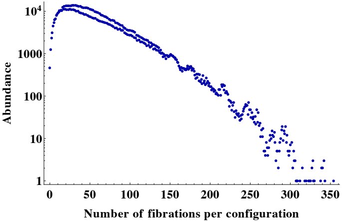
Multiplicity of fibrations

- Point of interest: When such manifolds have fibrations, they generically do not have just one...
- In fact there can be **many**
- E.g.

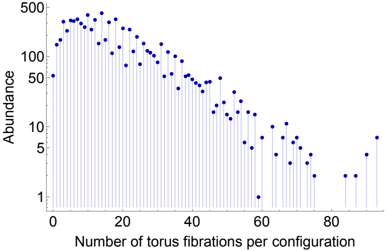
1) $\left[\begin{array}{c|ccc} \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right]$

2) $\left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \end{array}\right]$

- Thus far: "Obvious" fibrations



CICY 4-folds (Gray, et al 1405.2073)($\sim 10^8$ fibrations)



CICY 3-folds ($\sim$ 140, 000 fibrations)

- Result: We scanned for obvious fibrations in CICY 3-fold dataset ($7,890$ manifolds) and found $139,597$ genus one fibrations, $30,974$ K3-fibrations and $208,987$ nested combinations.
- But...no guarantee that obvious fibrations are the whole story. Need to be able to characterize fibrations independent of algebraic description of the manifold.
- It is possible to systematically characterize elliptic fibrations by characterizing the **base**: $\pi : X \to B$.

  Conjecture (Kollár): Let X be a Calabi-Yau n-fold. Then X is genus-1 fibered iff there exists a $(1,1)$-class D in $H^2(X,\mathbb{Q})$ such that $(D \cdot C) \geq 0$ for every algebraic curve $C \subset X$, $(D^{dim(X)}) = 0$ and $D^{dim(X)-1} \neq 0$.

  (proven for CY 3-folds (Wilson, Oguiso) with D effective or $D \cdot c_2(X) \neq 0$, evidence for $n > 3$)(Note: the fiber class $= D^2$).

Enumerating fibrations

- Can *all* genus-one fibrations in a dataset be enumerated? Given any dataset of CY 3-folds, can search for divisors satisfying the criteria, but need the following:
  1. Explicit control of all divisors $\longrightarrow$ Kähler and Mori cones of X
  2. Intersection numbers of X
- How would such a fibration count compare to "obvious" count? Once again, start with the simplest dataset of manifolds $7,890$ CICY 3-folds (Candelas, et al).
- Developed the tools to classify fibrations in CICYs via algorithmic replacement of CICYs with different algebraic descriptions, Gromov-Witten computations, etc.

Results:

1. Scanned over all 7890 CICY 3-folds in augmented dataset (with favorable replacements)
2. For favorable CICYs in which the Kähler cone strictly descends from the ambient product of projective spaces, we find (# of obvious fibrations = exhaustive list of fibrations). Otherwise, can find many more...
3. $> 99.4\%$ of CICY 3-folds are genus one fibered. All $\mathbb{E}$-fibered for $h^{1,1} > 4$.
4. Classify $377,559$ fibrations for the set of $4,957$ Kähler favorable manifolds.
5. Many non-obvious fibrations: E.g. anti-canonical hypersurface in $dP_7 \times dP_7$ gives rise to $15,878$ genus one fibrations. Can group into families based on birational equivalence.
6. For one manifold we find a parametrically infinite set of distinct elliptic fibrations $\Rightarrow (h^{1,1}, h^{2,1}) = (19, 19)$ manifold (Oguiso, Aspinwall/Gross).
7. These results extend to CYs with $\pi_1(X) \neq 0$, gCICYs, etc.

What about the physics of multiple elliptic fibrations?

- E.g. in 6D: M-theory on $Y_3 \simeq$ F-theory on $Y_3 \times S^1$
- Different fibrations of Y_3 will lead to different F-theory vacua, all with the same M-theory limit (i.e. Coulomb branch)
- Example with $h^{1,1} = 7$, $h^{2,1} = 26$:

$$X_3 = \left[\begin{array}{c|ccccccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ \mathbb{P}^2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \mathbb{P}^2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

E_1: $U(1)^5$
$V = 5,\ T = 0,\ H = 278$

E_2: $SU(2) \times U(1)^3$
$V = 6,\ T = 1,\ H = 250$

E_3: $SU(2) \times U(1)^3$
$V = 6,\ T = 1,\ H = 250$

E_4: $U(1)^4$
$V = 4,\ T = 1,\ H = 248$

E_5: $SU(2) \times U(1)^3$
$V = 6,\ T = 1,\ H = 250$

E_6: $SU(2)^2 \times U(1)^3$
$V = 9,\ T = 0,\ H = 282$

E_7: $SU(2) \times U(1)^4$
$V = 7,\ T = 0,\ H = 280$

E_8: $SU(2) \times U(1)^4$
$V = 7,\ T = 0,\ H = 280$

E_9: $SU(2) \times U(1)^4$
$V = 7,\ T = 0,\ H = 280$

$5D$ M-theory Limit
with
$V^M = 6,\ H^M = 27$

Heterotic/F-theory Duality

Based on 8-dimensional correspondence (Vafa):

$$\text{Heterotic on } \; \pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1} \quad \Leftrightarrow \quad \text{F-theory on } \; \pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1}$$

Leads to nested fibrations (w/ sections)

$$\begin{array}{ccccc} & Y_{n+1} & \xrightarrow{\mathbb{E}} & \mathcal{B}_n & \\ K3 & \downarrow & & \downarrow & \mathbb{P}^1 \\ & B_{n-1} & \xleftrightarrow{=} & B_{n-1} & \end{array}$$

Heterotic geometry also includes Mumford (slope) stable bundles $V \to X_n$

- Let's explore the possibilities for multiple $K3$ fibrations within the the F-theory geometry, Y_{n+1}, and what it means for the heterotic duals...

- Case 1: Multiple $K3$-fibrations with distinct elliptic fibrations $\rightarrow$ Same story as M-/F- correspondence. Duality in 1D lower.
- Case 2: Multiple $K3$-fibrations with *the same* elliptic fibration

$$\begin{tikzcd} & \mathcal{B}_n \arrow[dl, "\rho_1", "\mathbb{P}^1"'] \arrow[dr, "\rho_2"', "\mathbb{P}^1"] & \\ B_{n-1} & & B'_{n-1} \end{tikzcd}$$

$\rightarrow$ identical effective heterotic theories on different geometries:

$$\pi_h : X_n \rightarrow B_{n-1} \qquad \text{and} \qquad \pi'_h : X'_n \rightarrow B'_{n-1}$$

- Case 3: One $K3$-fibration with multiple elliptic fibrations:

$$\begin{tikzcd} & Y_{n+1} \arrow[dl, "\pi_1", "\mathbb{E}"'] \arrow[dr, "\pi_2"', "\mathbb{E}"] & \\ \mathcal{B}_n \arrow[dr, "\mathbb{P}^1", "\rho_1"'] & & \mathcal{B}'_n \arrow[dl, "\mathbb{P}^1"', "\rho_2"] \\ & B_{n-1} & \end{tikzcd}$$

$\rightarrow$ distinct heterotic theories on the same CY geometry: $\pi_h : X \rightarrow B_{n-1}$

Examples

- "Case 3" Can describe $E_8 \times E_8 / SO(32)$ duality in 8D and in lower dimensions (e.g. $\mathbb{P}^1$ ($SO(32), E_8 \times E_8$ duality $\to \pi_f : Y_3 \to \mathbb{F}_4$)
- Simple example of "Case 2": $\pi_f : Y_3 \to \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$
- Morrison+Vafa: These multiple fibrations lead to 6D heterotic/heterotic duality of Duff, Minasian, Witten ($n_T = 1$)
- When $n_T > 1$ much more is possible, by inspecting multiple fibrations, we find many generalizations of DMW in 6D (e.g. geometry given by conic bundles not $\mathbb{P}^1$-bundles).

Novel dualities in 4-dimensions

- Consider multiple elliptic fibrations within
- The 4-dimensional heterotic theory is independent of which way "up" it's elliptic fiber is oriented. Suppose

$$\begin{tikzcd} & X_3 \arrow[dl, "\mathbb{E}"', "\pi_1"] \arrow[dr, "\mathbb{E}", "\pi_2"'] & \\ B_2 & & B_2' \end{tikzcd}$$

- Leads to F-theory on two distinct 4-folds:

$$\rho : Y_4 \xrightarrow{K3} B_2, \quad , \quad \rho' : Y_4' \xrightarrow{K3} B_2'$$

with the same effective physics!

- Can shed light on G-flux.
- Systematic study of F-theory duals can lead to important bounds on the structure of $\mathcal{M}_X(c(V))$ (Mod. space of stable sheaves on X) (LA + Taylor)

Conclusions and Future Directions

- We have completed the first complete classification/enumeration of genus-one fibrations in a set of CICY 3-folds.
- Evidence is accumulating that as $h^{1,1}$ increases, all known CY manifolds are elliptically fibered $\rightarrow$ generically with multiple fibrations.
- *Can this be established for all CY 3-folds? If so, may provide a tractable approach to finiteness...*
- Multiple fibrations can impact a huge array of dualities and shed light on new aspects of string/F-theory.