Umbral and Penumbral Moonshine

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Cheng, Duncan & JH 1204.2779, 1307.5793

Cheng, Duncan 1605.04480

Rayhaun, JH 1504.08179

Duncan, Rayhaun, JH to appear 18xx.yyyyy ?

Thanks to

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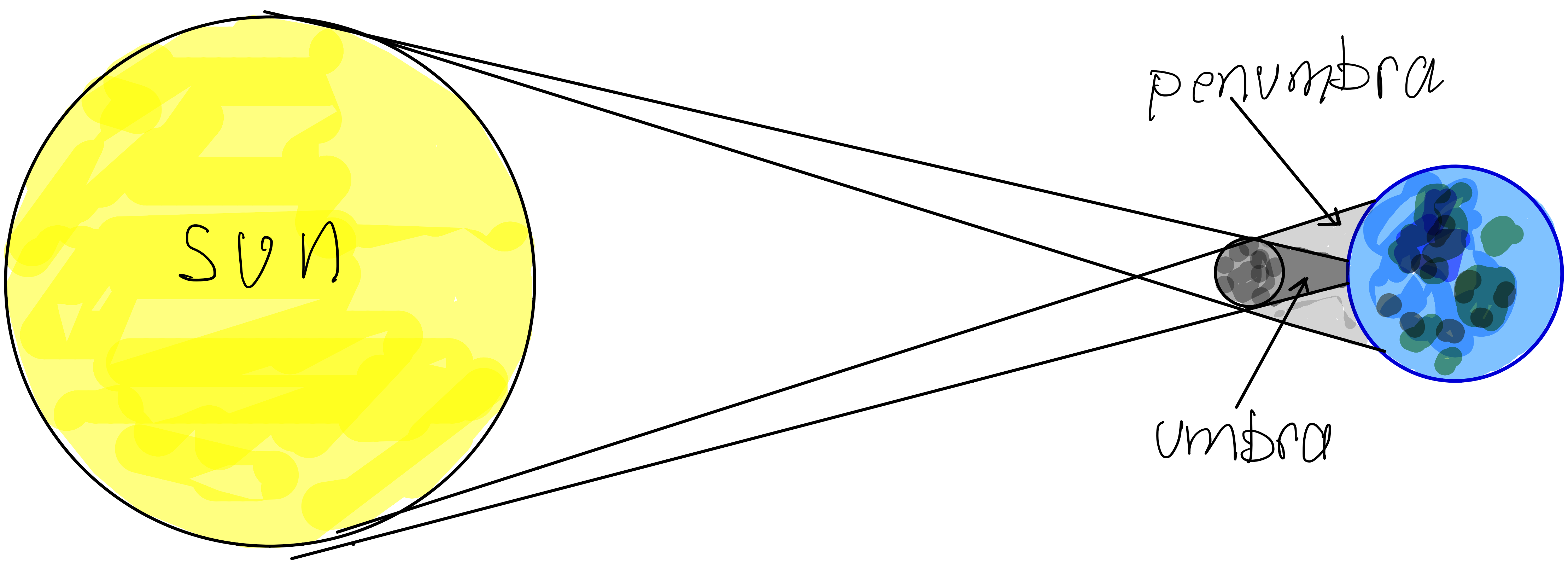
??? for arranging the moonshine talks to be within hours of the full moon in Okinawa.


sun
penumbra
umbra

Introduction

Modular forms play an important role in number theory and physics because they count things:

$$\frac{1}{\eta(\tau)^{24}} = \frac{q^{-1}}{\prod_n (1-q^n)^{24}} \in M^!_{-12}$$ States of open bosonic string

$$\Theta_{E_8}(\tau) = \sum_{p\in\Gamma_8} e^{\pi i \tau p^2} \in M_4$$ Points in E8 root lattice

$$\phi_{ell}(K3;\tau,z) \in J_{0,1}$$ BPS states on K3xS1

When the things being counted live in vector spaces which are representations of a finite group G we often say informally that the modular form “exhibits moonshine for G.”

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I will discuss two kinds of Moonshine with these properties:

1. Umbral: Started with EOT K3/M24 observation (2010). Generalized to other Jacobi forms and groups classified by Niemeier lattices (Cheng, Duncan, H).

2. Penumbral: Started with Thompson moonshine (H, Rayhaun), generalized to other skew-holo Jacobi forms (Duncan, H, Rayhaun, to appear).

Common element of Monstrous, Umbral and Penumbral moonshine: Role of genus zero groups

A class of genus zero subgroups of SL(2,R):

$$n + e, f, \cdots$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$$c = 0 \bmod n$$

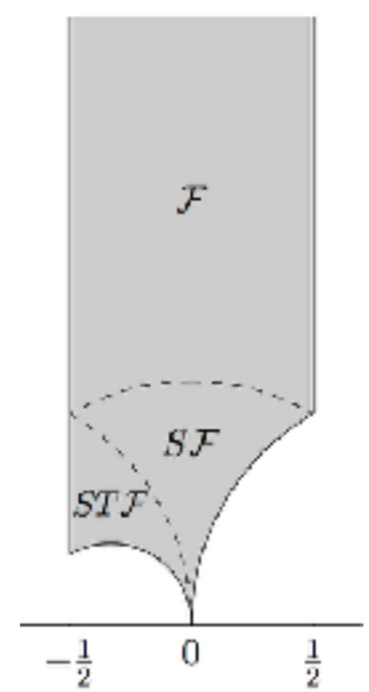
Atkin-Lehner

Fricke: $n \in e, f, \cdots$ or non-Fricke otherwise

$$\tau \to -1/n\tau$$

$$Z(2B, 1; \tau) = \left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24} + 24$$

$\Gamma_0(2)\backslash\mathcal{H}$



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Govern all twists and twines of Monstrous moonshine.

Classify cases of umbral and penumbral moonshine.

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Given $g, h \in \mathbb{M} \quad gh = hg \quad (\pi_1(T^2) \to \mathbb{M})$

$$\text{twine } h \boxed{\tau} = Z(g,h;\tau) = \mathrm{Tr}_{V_g} h q^{L_0 - c/24}$$

g
twist

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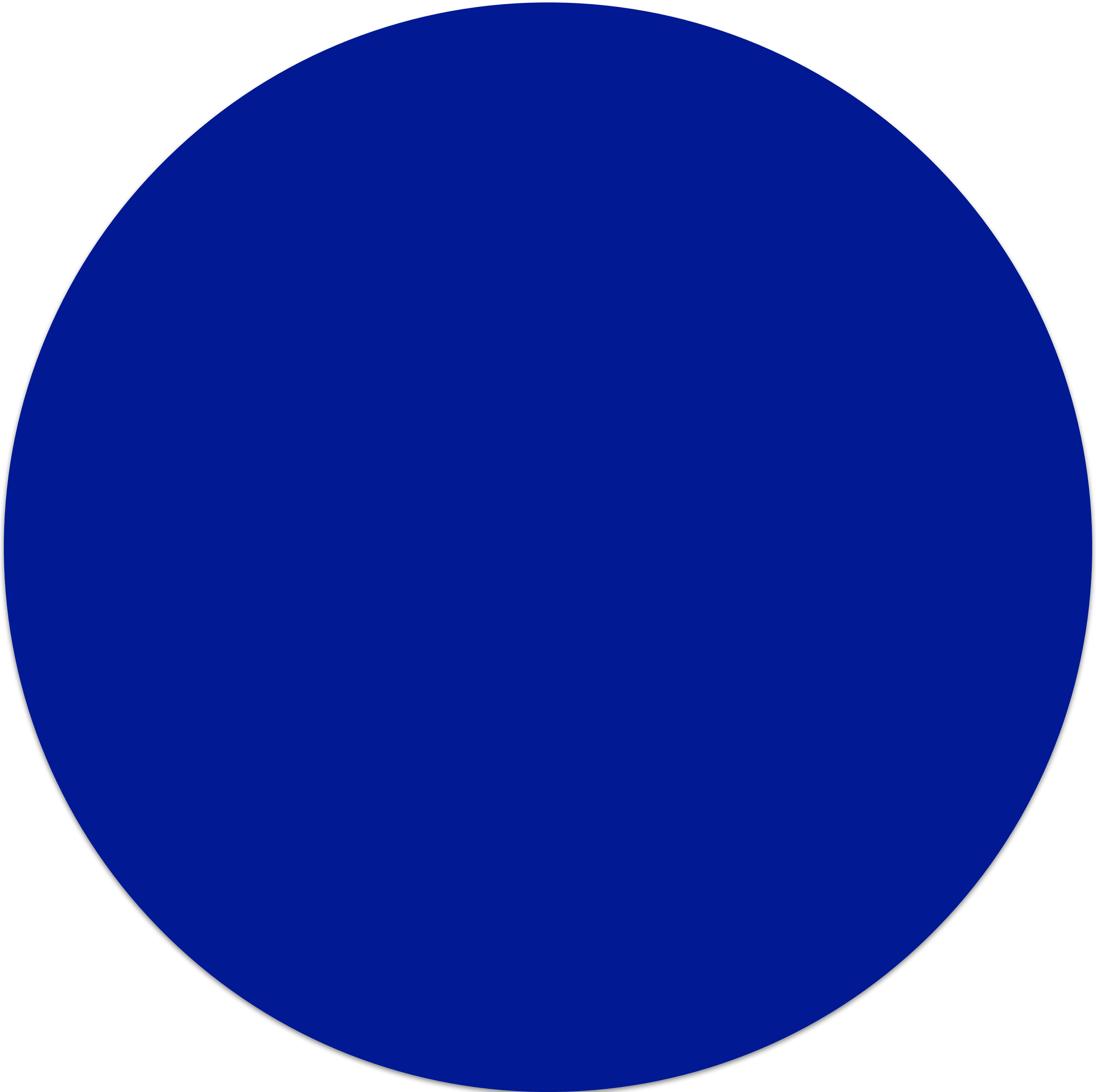
In (generalized) Monstrous Moonshine these are all genus zero functions (Conway, Norton, Queen, Borcherds, Carnahan)

Ogg: for p prime $p+p$ is genus zero precisely when p divides the order of the Monster.

Umbral Moonshine



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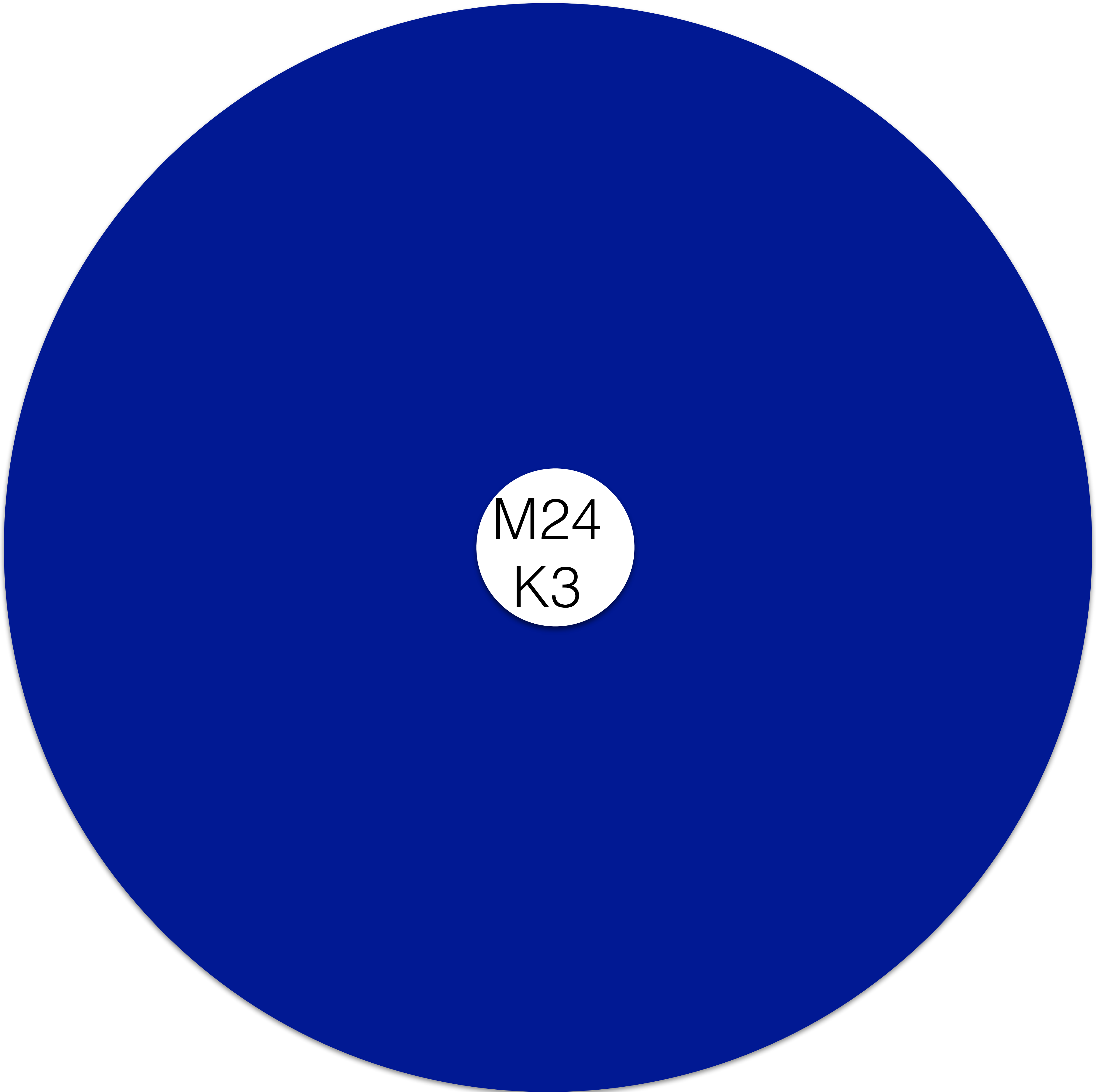

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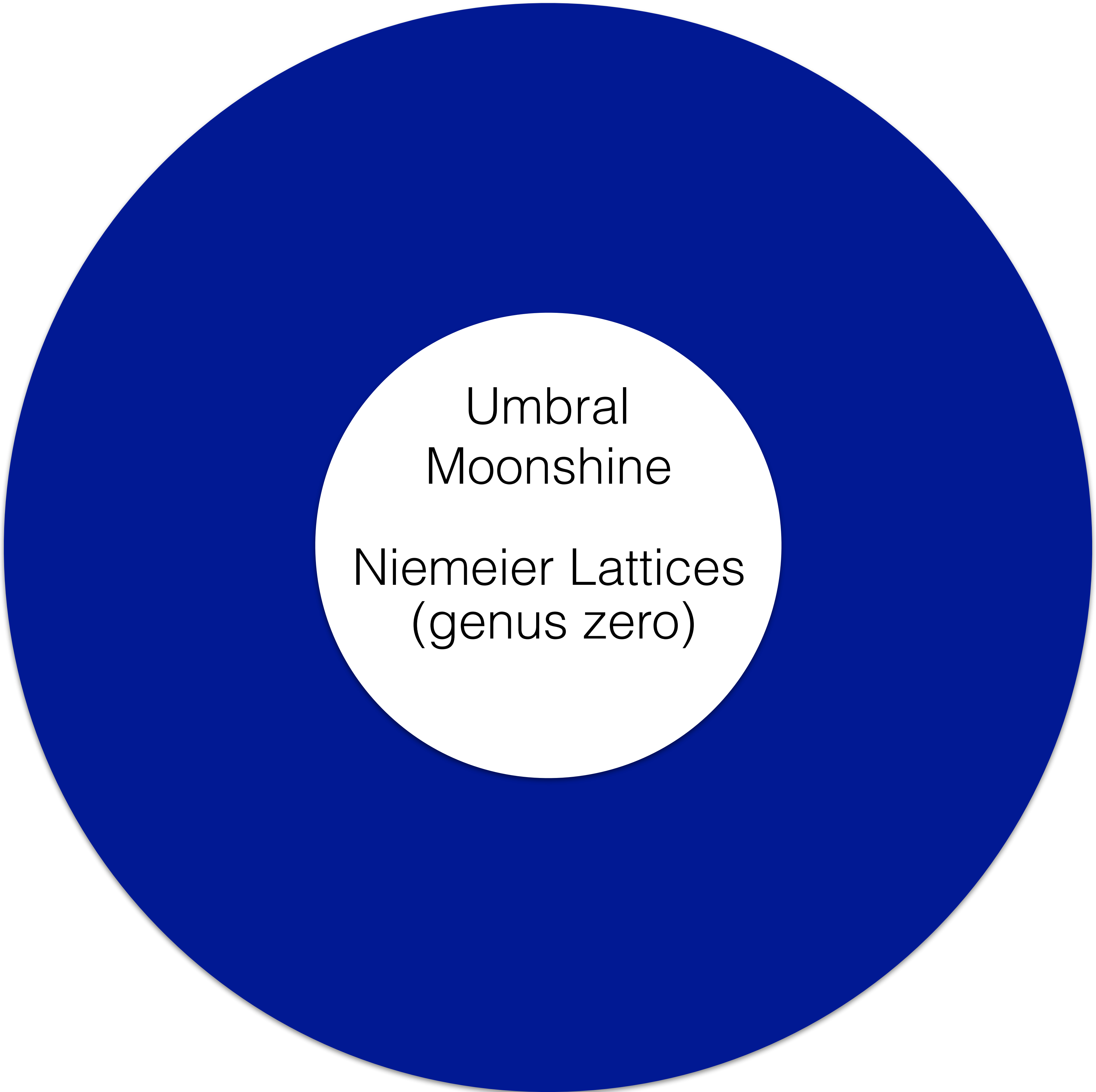


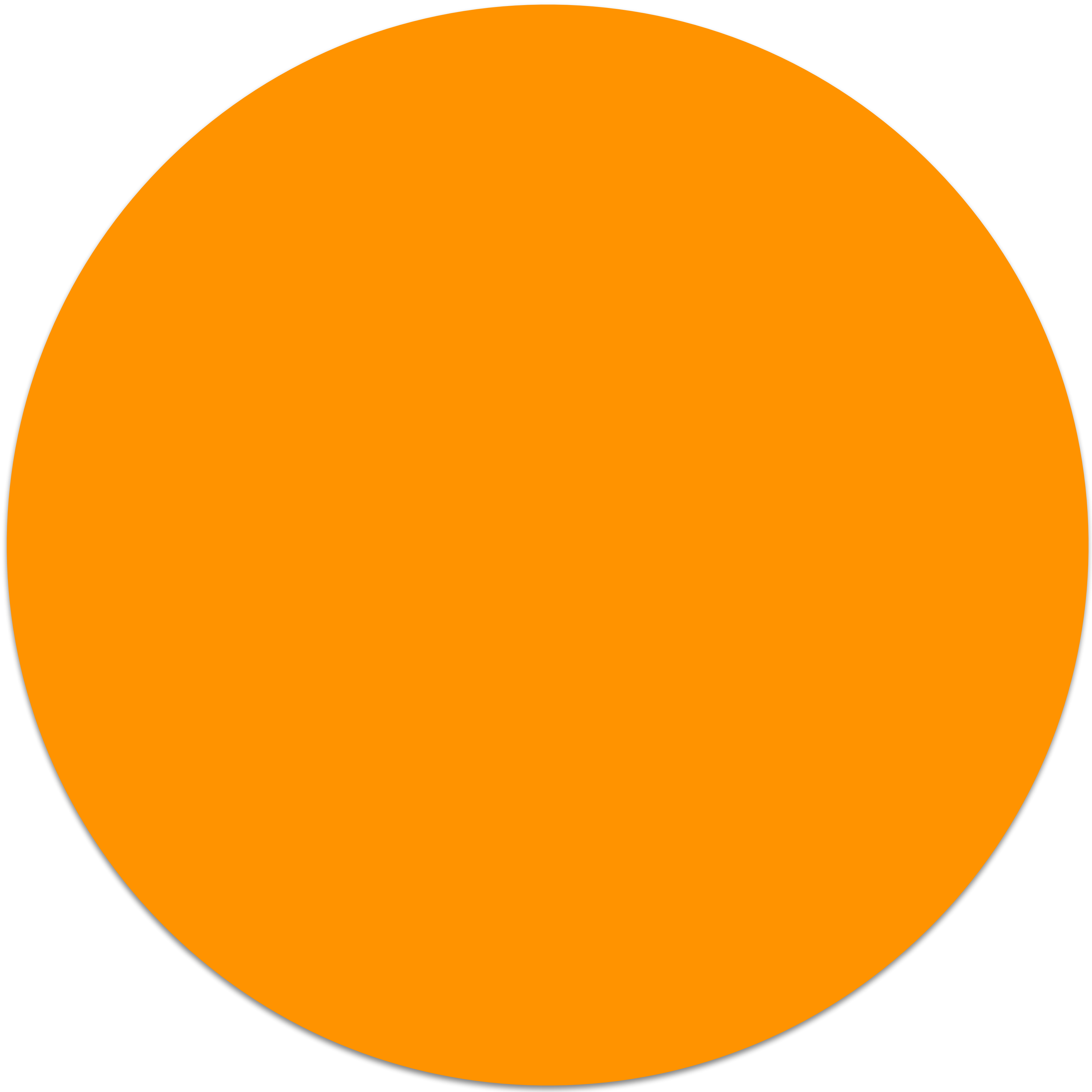
Umbral Moonshine

optimal mock Jacobi theta

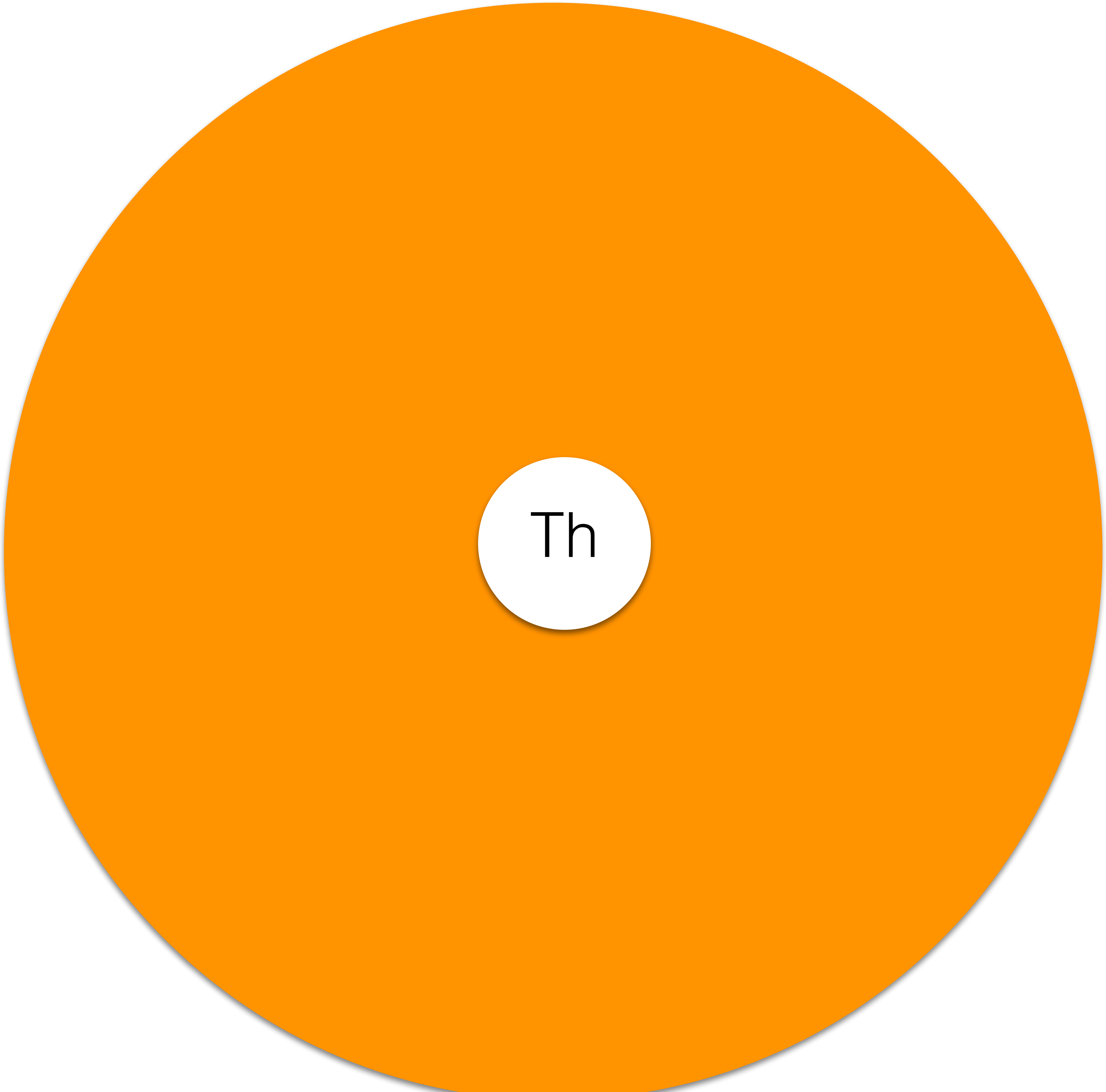
39 non-Fricke genus zero groups

(Cheng&Duncan)

Penumbral Moonshine



Penumbral Moonshine



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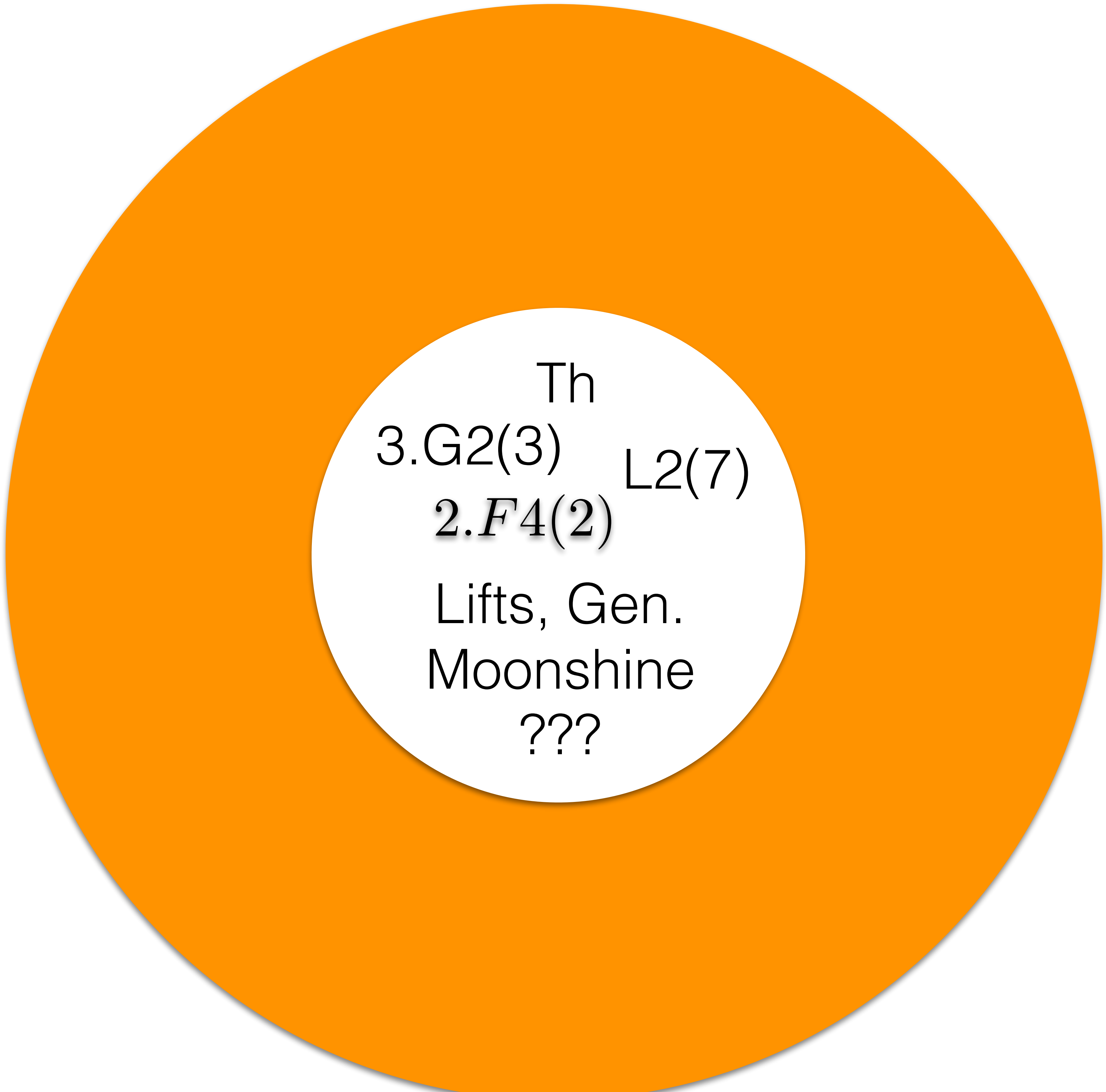

Penumbral Moonshine

Two Parallel Worlds of Weight 1/2 Moonshine

	Umbral	Penumbral
Modular objects	Optimal weight 1/2, mock Jacobi forms	Optimal weight 1/2 modular skew-holo Jacobi forms
Genus zero groups	non-Fricke	Fricke
Moonshine Groups	M24, Aut(Niemeier)	Groups of generalized moonshine? Lattices?
Explicit Constructions	For several, not uniform	None yet
Physics Connections	K3 elliptic genus	BPS counting functions at attractor points and Lifts (S. Harrison)

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What does it all mean?

What is a skew-holomorphic Jacobi form?

Eichler-Zagier Jacobi form: $\phi \in J_{1,m}$

$$\phi(\tau, z) = \sum_{r \bmod 2m} h_r(\tau)\theta_{m,r}(\tau, z) \qquad \Big(\theta_{m,r}(\tau, z) = \sum_{n=r \bmod 2m} q^{n^2/4m} y^n\Big)$$

Weight 1/2, these transform under a double cover of the modular group "Weil representation of the metaplectic group"

$$\theta_{m,r} \sim M_{1/2}(\rho_m) \qquad h_r \sim M_{1/2}(\overline{\rho_m})$$

Skew-holo Jacobi: $\phi_{1,m} \in J^{sk}_{1,m}$ $\quad h_r$ antiholomorphic in τ .

Jacobi: $\quad h_r \sim M_{1/2}(\overline{\rho_m}) \simeq J_{1,m}$

Skew-holo Jacobi: $\quad \overline{h_r} \sim M_{1/2}(\rho_m) \simeq J^{sk}_{1,m}$

Examples at m=1 ($f(\tau) = h_0(4\tau) + h_1(4\tau)$)

$$f_0 = \theta(\tau) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + O(q^{25})$$
$$f_3 = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + O(q^9)$$
$$f_4 = q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + O(q^9)$$
$$f_7 = q^{-7} - 4119q + 8288256q^4 - 52756480q^5 + 5734772736q^8 + O(q^9)$$
$$\cdots$$

(Borcherds, Zagier)

Skew-holo Jacobi forms in strings/BPS counting will be discussed by S. Harrison

What are the non-Fricke and Fricke genus 0 groups?

39 non-Fricke genus zero groups

Umbral

2	10	18	30+3,5,15
3	10+2	18+2	30+6,10,15
4	10+5	18+9	33+11
5	12	20+4	36+4
6	12+3	21+3	42+6,14,21
6+2	12+4	22+11	46+23
6+3	13	24+8	60+12,15,20
7	14+7	25	70+10,14,35
8	15+5	28+7	78+6,26,39
9	16	30+15	

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84 Fricke genus zero groups

1	17+17	31+31	51+3,17,51
2+2	18+2,9,18	32+32	54+2,27,54
3+3	18+18	33+33	55+5,11,55
4+4	19+19	34+34	56+7,8,58
5+5	20+4,5,20	35+5,7,35	59+59
6+2,3,6	20+20	35+35	60+3,4,5,12,15,20,60
6+6	21+3,7,21	36+4,9,36	60+4,15,60
7+7	21+21	36+36	62+2,31,62
8+8	22+2,11,22	38+38	66+2,3,11,6,22,33,66
9+9	23+23	39+3,13,39	66+6,11,66
10+2,5,10	24+3,8,24	39+39	69+3,23,69
10+10	24+24	41+41	70+2,5,7,10,14,35,70
11+11	25+25	42+2,3,7,6,14,21,42	71+71
12+4,3,12	26+2,13,26	42+3,14,42	78+2,3,13,6,26,39,78
12+12	26+26	44+4,11,44	87+3,29,87
13+13	27+27	45+5,9,45	92+4,23,92
14+2,3,7	28+4,7,28	46+2,23,46	94+2,47,94
14+14	29+29	47+47	95+5,19,95
15+3,5,15	30+2,3,5,6,10,15,30	49+49	105+3,5,7,15,21,35,105
15+15	30+2,15,30	50+2,25,50	110+2,5,11,10,22,55,110
16+16	30+5,6,30	50+50	119+7,17,119

What role does genus zero play?

$$j(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

Rademacher: $j(\tau)$ can be obtained by averaging q^{-1} over $SL_2(\mathbb{Z})$ modulo its stabilizer, but one must regularize. This can be generalized to other genus zero hauptmoduls (Knopp, Duncan&Frenkel)

$$f_0(\tau) = \mathrm{Reg}\left(\sum_{\gamma \in \Gamma_{inv}/\Gamma_0} f_0^{\mathrm{polar}}|_\gamma \right)$$ holomorphic, not obviously modular

Extended to other weights and multiplier systems by Knopp, Niebur, Bringmann-Ono, Cheng-Duncan.

Farey Tales (Dijkgraaf, Maldacena, Moore, Verlinde, Manschot,…):

An interpretation as a sum over asymptotic AdS_3 geometries with T^2 boundary in context of BH counting

Obstruction to modularity of Rademacher sums:

Modular Mock Cusp (shadow)

$$0 \to M_k^! \to \mathbb{M}_k \to S_{2-k} \to 0 \quad \text{on} \quad \Gamma < SL(2,\mathbb{R})$$

Main Point: At weight zero, Rademacher gives modular functions when there are no weight two cusp forms and mock modular forms otherwise.

Weight 2: $s(\tau)d\tau = s(\tau')d\tau'$ holo 1-form on $\Gamma\backslash\mathbb{H}$

$\Longrightarrow$ genus $(\Gamma\backslash\mathbb{H})$ >0

$$0 \to M_0(m) \to \mathbb{M}_0(m) \to S_2(m) \to 0$$

Shimura, Shintani, Brunier, Ono, Skoruppa, Zagier

$$0 \to M_{1/2}(\rho_m) \to \mathbb{M}_{1/2}(\rho_m) \to S_{3/2}(\bar{\rho}_m) \to 0$$

When Rademacher gives mock modular forms, they have rational coefficients only when the shadows are theta functions in one variable

Cheng-Duncan: Genus zero classification of optimal mock Jacobi theta functions (including Umbral forms)

Duncan, H, Rayhaun: Genus zero classification of possible skew-holo Jacobi forms of penumbral moonshine

What are some examples of penumbral moonshine?

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Optimal skew-holo Jacobi forms of penumbral moonshine

$$\phi^{(m,D_0)} \in J^{sk}_{1,m} \quad h_s = 2q^{D_0/4m} + O(1) \quad \text{with} \quad D_0 = s^2 \bmod 4m$$

$$h_r = O(1), \; r \neq s$$

For a given D_0 such forms exist for finitely many m (lambency) indexed by genus zero (Fricke) subgroups of $SL_2(\mathbb{R})$.

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Thompson: $$\mathcal{F}_3(\tau) = 2f_3(\tau) + 248\theta(\tau) = \sum_m c(m)q^m$$

$c(k)$	Decomposition
$c(-3)$	$2 \cdot {}^1V_1$
$c(0)$	${}^{248}V_2$
$c(4)$	${}^{27000}V_4 \oplus {}^{27000}V_5$
$-c(5)$	${}^{85995}V_9 \oplus {}^{85995}V_{10}$
$c(8)$	${}^{1707264}V_{17} \oplus {}^{1707264}V_{18}$
$-c(9)$	${}^{4096000}V_{22} \oplus {}^{4096000}V_{23}$
$c(12)$	$2 \cdot {}^{44330496}V_{40}$
$-c(13)$	$2 \cdot {}^{91171899}V_{46} \oplus {}^{779247}V_{14} \oplus {}^{779247}V_{15}$

$$\boxed{D_0 = -3}$$

The first few allowed m values are 1,3,7,13,19,21,31.

m=1 is the Thompson moonshine example.

m=3 leads to moonshine for $3.G_2(3)$ which is related to the centralizer of an element of order 3 in Th.

m=7 leads to moonshine for $L_2(7)$ which is related to the centralizers of an element of order 7 in Th.

m=13,19,31 are the other prime values m dividing the order of Th. For 19,31 the centralizers are Abelian cyclic groups of these orders.

$$\boxed{(m, D_0) = (1, -4)}$$

$$\mathcal{F}_0 = q^{-1} - 492 + 2 \times \underline{142884q} + 2 \times 18473000q^2 +$$

$$\mathcal{F}_1 = 2 \times \underline{565760q}^{5/4} + 2 \times 51179520q^{9/4} + \cdots$$

Moonshine for 2.F4(2)

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Moonshine for Baby Monster as realized in Hohn's c=23 1/2 CFT

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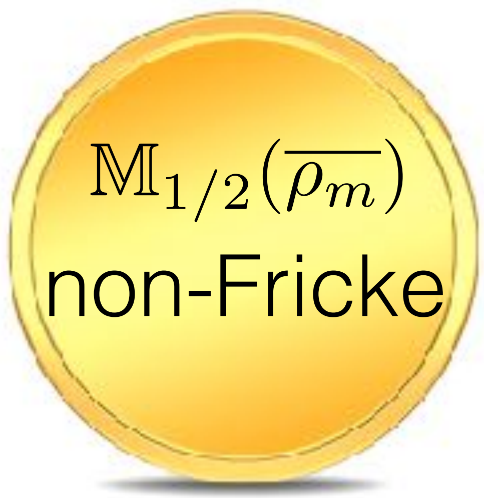
$(m, D_0) = (6, -23)$

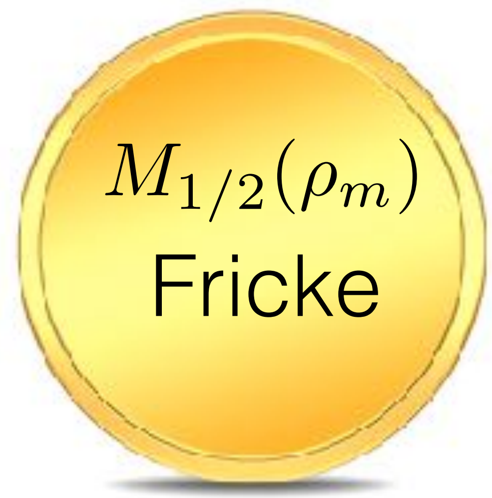
$$\eta J(\tau) = q^{-23/24} - q^{1/24} + 196883q^{25/24} + 21296876q^{49/24}$$

Moonshine for Monster at weight 1/2 (decomposition into Virasoro characters)

What does it all mean?

Mathematically umbral and penumbral moonshine look like two sides of a single coin:





There are many connections to generalized Monstrous Moonshine via Groups and Lifts.

For each prime p dividing $|\mathbb{M}|$ there is a weakly holomorphic weight 1/2 form $f^{(p)} = \sum_n c^{(p)}(n) q^n$

$$Z(g,1,\tau) = q^{-c} \prod_{n=1}^{\infty} (1-q^n)^{c(n^2)}$$

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Does string theory provide an understanding of the vector spaces and representations being counted in these new examples of moonshine?

Thank You

(and please enjoy the moonshine tonight)

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