Inside the walls of positive geometry: the space of consistent QFTs

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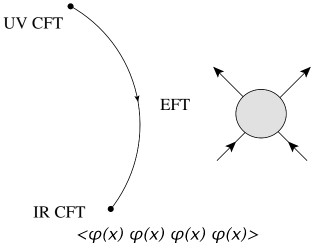
String 2018 @OIST-June-26-2018

We want to understand what “is” the space of consistent QFT

$$\text{consistent} \equiv \text{Unitarity},\ \text{Locality},\ \text{Spacetime symmetries}$$

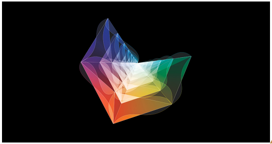
We explore the space through the lens of physical observables



potential to explore new principles governing the “is”

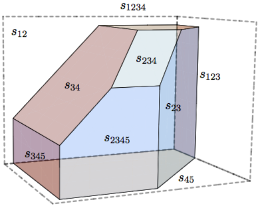
In the study of S-matrix for specific theories, locality and unitarity $\rightarrow$ emergent from Positive geometry

$N = 4SYM$:



Arkani – hamed, Trnka

ϕ^3 :



Arkani – Hamed, Bai, He, Yan

One might be tempted (ambitious) to ask:

Is positive geometry the underlying property of general QFTs?

For a long time positivity IS unitarity

- Positivity in the OPE:

$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle = \sum_i \mathfrak{p}_i K_{\Delta_i,\ell_i}(z,\bar{z}), \quad \mathfrak{p}_i > 0$$

- Optical theorem:

$$Dis[M_4(s,0)] = E_{cm}^2\sigma > 0$$

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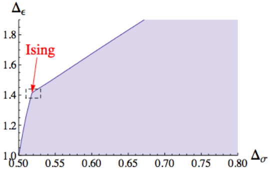
via crossing El-Showka, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi



- Optical theorem:

$$Dis[M_4(s,0)] = E_{cm}^2\sigma > 0$$

via the eyes of higher-dimension operators $a(\partial\phi)^4$ Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi

We should expect more: these are special functions, constrained by factorization and symmetries

- : CFTs:

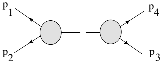
$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle = \sum_i \mathfrak{p}_i g_{\Delta_i,\ell_i}(z,\bar{z}), \quad \mathfrak{p}_i > 0$$

  Symmetries constrain

$$(z^2(1-z)\partial_z^2 - z^2\partial_z)g_{\Delta,\ell} = \Delta(\Delta-1)g_{\Delta,\ell}$$

- QFTs:

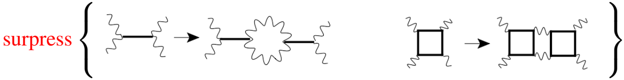
$$Dis[M_4(s,t)] = \sum_i \mathfrak{p}_i G^{\alpha}_{\ell_i}(\cos\theta)$$

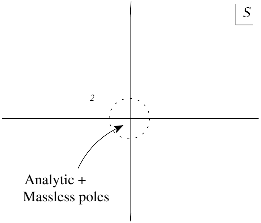


$$= p_{12}^{\mu_1}\cdots p_{12}^{\mu_\ell} P_{\mu_1\cdots\mu_\ell\nu_1\cdots\nu_\ell} p_{34}^{\mu_1}\cdots p_{34}^{\mu_\ell} = G^{\alpha}_{\ell}\left(1+\frac{2t}{m^2}\right)$$

The geometric constraint for general QFT

Consider general QFT whose UV completion is weakly coupled (in M_{pl}),





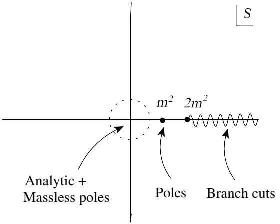
$$M^{IR}(s,t) = \{poles\} + \sum_{k,i} g_{k-i,i} s^{k-i} t^i$$

Different QFTs (standard model) leads to different $\{g_{i,j}\}$

Why might the space be non-trivial?

The geometric constraint for general QFT

Why is the space non-trivial (set $D = 4$ $G_\ell^\alpha \to P_\ell$)?



$$M(s,t) = -\sum_a \mathfrak{p}_a P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)\left(\frac{1}{s-m_a^2}\right)$$

$$= \sum_{k,q}\sum_a \mathfrak{p}_a \frac{1}{m_a^{2k+2}} u^q_{k,\ell_a} s^{k-q} t^q$$

so we have

$$\sum_{k,q} g_{k-q,q} s^{k-q} t^q = \sum_{k,q}\left(\sum_a \mathfrak{p}_a \frac{1}{m_a^{2k+2}} u^q_{k,\ell_a}\right) s^{k-q} t^q$$

The geometric constraint for general QFT

Why is the space non-trivial?

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Organizing the higher dimension operators as

$$\begin{array}{cccccc}
 & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\
t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\
t^1 & & g_{0,1} & g_{1,1} & g_{2,1} & \cdots \\
t^2 & & & g_{0,2} & g_{1,2} & \cdots \\
t^3 & & & & g_{0,3} & \cdots
\end{array}$$

The geometric constraint for general QFT

Why is the space non-trivial?

$$\sum_{k,q} g_{k-q,q} s^{k-q} t^q = \sum_{k,q} \left(\sum_a \mathfrak{p}_a \frac{1}{m_a^{2k+2}} u^q_{k,\ell_a} \right) s^{k-q} t^q$$

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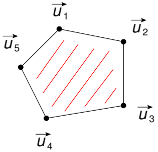
Take $k = 2$ (dimension 8 operators)

$$\vec{g}_2 = \begin{pmatrix} g_{2,0} \\ g_{1,1} \\ g_{0,2} \end{pmatrix} \in \sum_a \mathfrak{p}'_a \begin{pmatrix} u^0_{2,\ell_a} \\ u^1_{2,\ell_a} \\ u^2_{2,\ell_a} \end{pmatrix} = \sum_a \mathfrak{p}'_a \vec{u}_{2,\ell_a} \quad \mathfrak{p}'_a > 0$$

The coefficients must live in the convex hull of the vectors $\vec{u}_{2,\ell}$, i.e. the inside of a polytope.

Polytope 101



$$\vec{A} = \frac{\sum_{i=1}^{5} m_i \vec{u}_i}{\sum_{i=1}^{5} m_i}$$

Write things projectively:

$$A^l = \begin{pmatrix} 1 \\ \vec{A} \end{pmatrix}, \quad U_i^l = \begin{pmatrix} 1 \\ \vec{u}_i \end{pmatrix}$$

The convex hull is the inside of the polygon

$$A^l = w_1 U_1^l + \cdots + w_n U_n^l, \quad w_i > 0, \; \sum_{i=1}^{n} w_i = 1$$

The inside is determined by $Det[A, U_i, U_j] > 0$

The facet structure is determined by $Det[U_i, U_j, U_k]$

Polytope 101

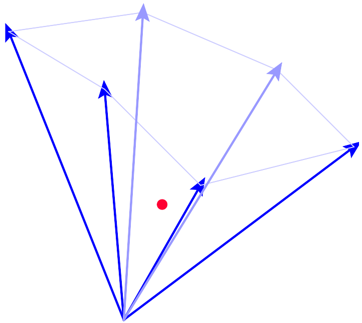
If $\vec{u}_{k,\ell}$ for EFT are just random vectors, our geometric problem becomes hopeless rapidly:

Let's say given n vectors $\vec{u}$, to compute the region of the polytope we need to

- Determine which one of these $\vec{u}$s are vertices
- Amongst the vertices, determine all the set that constitute boundary facets

The boundaries are (for $D = 2$) $\{U_a, U_b\}$

$$Det[U_i, U_a, U_b] \geq 0 \quad \forall i$$



The complexity is $\sim n^{d/2}$

But $\vec{u}_{k,\ell}$ are not random vectors!

Gegenbauer Positivity

The $\vec{u}_{k,\ell}$, arrises from Taylor expand

$$M(s,t) = \sum_a \mathfrak{p}_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2}$$

First define

$$P_\ell(1+x) = \sum_q v_{\ell,q} x^q$$

The vectors $\vec{v}_\ell = (v_{\ell,0},\ v_{\ell,1},\ v_{\ell,2},\cdots)$ are explicitly given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 \\ 0 & 0 & \frac{3}{2} & \frac{15}{2} & \frac{45}{2} & \frac{105}{2} & 105 & 189 \\ 0 & 0 & 0 & \frac{5}{2} & \frac{35}{2} & 70 & 210 & 525 \\ 0 & 0 & 0 & 0 & \frac{35}{8} & \frac{315}{8} & \frac{1575}{8} & \frac{5775}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{63}{8} & \frac{693}{8} & \frac{2079}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{231}{16} & \frac{3003}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{429}{16}. \end{pmatrix}$$

Gegenbauer Positivity

All v is positive!

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 \\ 0 & 0 & \frac{3}{2} & \frac{15}{2} & \frac{45}{2} & \frac{105}{2} & 105 & 189 \\ 0 & 0 & 0 & \frac{5}{2} & \frac{35}{2} & 70 & 210 & 525 \\ 0 & 0 & 0 & 0 & \frac{35}{8} & \frac{315}{8} & \frac{1575}{8} & \frac{5775}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{63}{8} & \frac{693}{8} & \frac{2079}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{231}{16} & \frac{3003}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{429}{16} \end{pmatrix}.$$

But there is more,

$$det[\vec{v}_{\ell_1}\, \vec{v}_{\ell_2} \cdots] > 0, \quad \text{for} \ell_1 > \ell_2 > \cdots$$

All ordered minors are positive!

Gegenbauer Positivity

$$det[\vec{v}_{\ell_1} \vec{v}_{\ell_2} \cdots] > 0, \quad \text{for} \ell_1 > \ell_2 > \cdots$$

All ordered minors are positive!

Tells us that the convex hull of $\{\vec{v}_\ell\}$ is a cyclic polytope

- All $\vec{v}_\ell$ are vertices
- The co-dimension 1 boundaries are known. For $\vec{v}_\ell = (v_{\ell,0}, \cdots, v_{\ell,q})$

$$q \in even \quad (i, i{+}1), \quad (i, i{+}1, j, j{+}1), \quad (i, i{+}1, \cdots, j, j{+}1)$$

$$q \in odd \quad (1, i, i{+}1), \quad (1, i, i{+}1 \cdots j, j{+}1), \quad (i, i{+}1, n), \quad (i, i{+}1 \cdots j, j{+}1, n)$$

Double positivity

But there is more ! $\vec{v}_\ell$ is not $\vec{u}_{k,\ell}$,

$$
\begin{aligned}
M(s,t) &= -\sum_a \mathfrak{p}_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2} \\
&= \sum_a \mathfrak{p}_a \frac{1}{m_a^2}\left(1+\frac{s}{m_a^2}+\frac{s^2}{m_a^4}+\cdots\right)_{locality}\left(v_{\ell_a,0}+v_{\ell_a,1}\frac{t}{m_a^2}+v_{\ell_a,2}\frac{t^2}{m_a^4}\cdots\right)_{unitarity}
\end{aligned}
$$

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For fixed mass-dimensions we indeed have

$$\begin{array}{cccccc} & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\ t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\ t^1 & & g_{0,1} & g_{1,1} & g_{2,1} & \cdots \\ t^2 & & & g_{0,2} & g_{1,2} & \cdots \\ t^3 & & & & g_{0,3} & \cdots \end{array}$$

$$\vec{g}_2 = \begin{pmatrix} g_{2,0} \\ g_{1,1} \\ g_{0,2} \end{pmatrix} \in \sum_a \mathfrak{p}'_a \vec{v}_a \quad \mathfrak{p}'_a > 0$$

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t^3 & & & & g_{0,3} & \cdots
\end{array}
$$

$$
\vec{g}_2 = \begin{pmatrix} g_{2,0} \\ g_{1,1} \\ g_{0,2} \end{pmatrix} \to Det[\vec{g}_2, \vec{v}_\ell, \vec{v}_{\ell+1}] > 0
$$

Double positivity

But there is more !

$$M(s,t) = -\sum_a \mathfrak{p}_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2}$$

$$= \sum_a \mathfrak{p}_a \frac{1}{m_a^2} \left(1+\frac{s}{m_a^2}+\frac{s^2}{m_a^4}+\cdots\right)_{locality} \left(v_{\ell_a,0}+v_{\ell_a,1}\frac{t}{m_a^2}+v_{\ell_a,2}\frac{t^2}{m_a^4}\cdots\right)_{unitarity}$$

But for fixed degree in t (scattering angle)

$$\begin{array}{ccccccc} & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\ t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\ t^1 & & g_{0,1} & g_{1,1} & g_{2,1} & \cdots \\ t^2 & & & g_{0,2} & g_{1,2} & \cdots \\ t^3 & & & & g_{0,3} & \cdots \end{array}$$

$$\begin{pmatrix} g_{0,1} \\ g_{1,1} \\ g_{2,1} \end{pmatrix} \in \sum_a \mathfrak{p}'_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \end{pmatrix} \qquad \mathfrak{p}'_a > 0$$

The vector is in the convex hull of points on the half-moment curve!

$$(1, x, x^2, \cdots, x^a), \quad x \in R^+$$

Double positivity

$$(1, x, x^2, \cdots, x^a), \quad x \in R^+$$

Organizing the couplings for fixed t power into the Hankel matrix ($g'_k \equiv g_{k,i}$)

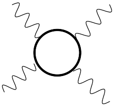
$$K(g') = \begin{pmatrix} 1 & g'_1 & \cdots & g'_{p-1} \\ g'_1 & g'_2 & \cdots & g'_p \\ \vdots & \vdots & \vdots & \vdots \\ g'_{p-1} & g'_p & \cdots & g'_{2p-2} \end{pmatrix},$$

If $\{g'_i\}$ lies in the convex hull of half-moment curves, then all minors of $K[g']$ is positive!

$$i \in even: \quad Det \begin{pmatrix} 1 & g'_1 & \cdots & g'_{\frac{i}{2}} \\ g'_1 & g'_2 & \cdots & g'_{\frac{i}{2}+1} \\ \vdots & \vdots & \vdots & \vdots \\ g'_{\frac{i}{2}} & g'_{\frac{i}{2}+1} & \cdots & g'_i \end{pmatrix} \geq 0, \qquad i \in odd: \quad Det \begin{pmatrix} g'_1 & g'_2 & \cdots & g'_{\frac{i+1}{2}} \\ g'_2 & g'_3 & \cdots & g'_{\frac{i+3}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ g'_{\frac{i+1}{2}} & g'_{\frac{i+3}{2}} & \cdots & g'_i \end{pmatrix} \geq 0$$

Double positivity

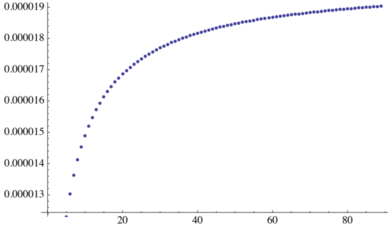
Consider the EFT of a scalar coupled to gravitons Congkao Wen, Wei-Ming Chen, Y-t



$$\Big|_{t=0} = \frac{\langle 23\rangle^4[14]^4}{M_{pl}^4}\left(\frac{1}{50400} + \frac{1}{17297280}\frac{s}{m^2} + \cdots\right)$$

$$= \frac{\langle 23\rangle^4[14]^4}{M_{pl}^4}\left(\sum_j 3\sqrt{\pi}4^{-2j-3}\frac{\Gamma[2j-1]}{(4+j)\Gamma[2j+\frac{7}{2}]}s^{j-1}\right)$$

Let's suppose we don't know the constant piece. The positivity of the Hankel matrix yields $\mathcal{O}(s^0) \geq 0.0000190301$



while $\frac{1}{50400} = 0.0000198413$

Double positivity

We see that the constraint from unitarity, locality and Lorentz invariance forces the EFT to live in a union of two positive geometries

$$
\begin{aligned}
M(s,t) &= -\sum_a \mathfrak{p}_a \frac{P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)}{s-m_a^2} \\
&= \sum_a \mathfrak{p}_a \frac{1}{m_a^2} \underbrace{\left(1+\frac{s}{m_a^2}+\frac{s^2}{m_a^4}+\cdots\right)_{locality}}_{(Conv[moment\ curve])} \underbrace{\left(v_{\ell_a,0}+v_{\ell_a,1}t+v_{\ell_a,2}\frac{t^2}{m_a^4}\cdots\right)_{unitarity}}_{(Conv[cyclic\ polytope])}
\end{aligned}
$$

The EFTHedron

The union of the two positive geometry leads to the a new generalization of polytopes: the EFThedron

- Polytopes: convex hull of vectors

$$\vec{Y} = \sum_i c_i \vec{v}_i, \quad c_i > 0$$

- Cyclic Polytopes: convex hull of vectors

$$\vec{Y} = \sum_i c_i \vec{v}_i, \quad c_i > 0$$

  where

$$\langle \vec{v}_{i_1}, \vec{v}_{i_2}, \cdots, \vec{v}_{i_n} \rangle > 0, \quad \text{for} \quad i_1 < i_2 < \cdots < i_n$$

- The EFThedron:

$$\vec{Y}_k = \sum_i c_{i,k} \vec{v}_i,$$

  where

$$\langle \vec{v}_{i_1}, \vec{v}_{i_2}, \cdots, \vec{v}_{i_n} \rangle > 0, \quad \text{for} \quad i_1 < i_2 < \cdots < i_n$$

  and

$$minors\,\{K[c_{i,k}]\} > 0$$

The EFTHedron

For general scalar EFT with color ordering (large-N YM),[1] the space of couplings $\{g_{k,q}\}$, consistent with Unitarity, Locality and Lorentz symmetry is given by

$$\vec{g}_k \in \sum_i c_{i,k} \vec{v}_i ,$$

In other words, the IR avatar of Unitarity, Locality and Lorentz invariant UV completion is the The EFTHedron

[1] More on this in 2 slides

The EFTHedron

- The EFThedron:

$$\vec{Y}_k = \sum_i c_{i,k} \vec{v}_i,$$

  where

$$\langle \vec{v}_{i_1}, \vec{v}_{i_2}, \cdots, \vec{v}_{i_n} \rangle > 0, \quad \text{for} \quad i_1 < i_2 < \cdots < i_n$$

  and

$$minors \{ K[c_{i,k}] \} > 0$$

- The Amplitudhedron:

$$\vec{Y}_k = \sum_i c_{i,k} \vec{v}_i,$$

  where

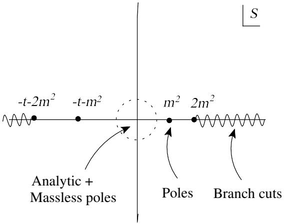
$$\langle \vec{v}_{i_1}, \vec{v}_{i_2}, \cdots, \vec{v}_{i_n} \rangle > 0, \quad \text{for} \quad i_1 < i_2 < \cdots < i_n$$

  and

$$c_{i,k} \in Gr_+(k, n)$$

The EFTHedron in the real world

Including the u-channel contribution:



$$M(s,t) = -\sum_a \mathfrak{p}_a P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)\left(\frac{1}{s-m_a^2}+\frac{1}{u-m_a^2}\right)$$

$$\rightarrow M(z,t) = -\sum_a \mathfrak{p}_a P_{\ell_a}\left(1+\frac{2t}{m_a^2}\right)\left(\frac{1}{-\frac{t}{2}-z-m_a^2}+\frac{1}{-\frac{t}{2}+z-m_a^2}\right)$$

Upon Taylor expansion we again have

$$\sum_{k-q\in even,\,q}\sum_a p_a\left[\frac{1}{m_a^{2(k+1)}}u_{\ell_a,k,q}\right]z^{k-q}t^q$$

where the now the vectors $\vec{u}_{\ell,k}$ are given as a k-dependent linear combination of the Gegenbauer vectors

$$u_{\ell,k,q} = \sum_{a+b=q}(-)^a\frac{(k-q+1)_a}{a!}2^{b-a}v_{\ell,b}$$

The EFTHedron in the real world

$$u_{\ell,k,q} = \sum_{a+b=q} (-)^a \frac{(k-q+1)_a}{a!} 2^{b-a} v_{\ell,b}$$

This in principle destroys any positivity. For example:

$$\begin{aligned}
&Det\begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^2_{\ell_1} - \frac{3}{4}v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^4_{\ell_1} - \frac{1}{4}v^3_{\ell_1} + \frac{1}{16}v^2_{\ell_1} - \frac{1}{64}v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} \\
&= Det\begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^2_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^4_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} - \frac{3}{4} Det\begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^4_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} - \frac{1}{32} Det\begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^2_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} \\
&- \frac{1}{4} Det\begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^2_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^3_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} + \frac{3}{16} Det\begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^3_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} + \cdots
\end{aligned}$$

The EFTHedron in the real world

$$u_{\ell,k,q} = \sum_{a+b=q} (-)^a \frac{(k-q+1)_a}{a!} 2^{b-a} v_{\ell,b}$$

This in principle destroys any positivity. For example:

$$Det \begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^2_{\ell_1} - \frac{3}{4} v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^4_{\ell_1} - \frac{1}{4} v^3_{\ell_1} + \frac{1}{16} v^2_{\ell_1} - \frac{1}{64} v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix}$$
$$= Det \begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^2_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^4_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} - \frac{3}{4} Det \begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^4_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} - \frac{1}{32} Det \begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^2_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix}$$
$$- \frac{1}{4} Det \begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^2_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^3_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} + \frac{3}{16} Det \begin{pmatrix} v^0_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^1_{\ell_1} & \{\ell_2\} & \{\ell_3\} \\ v^3_{\ell_1} & \{\ell_2\} & \{\ell_3\} \end{pmatrix} + \cdots$$

Yet above a critical spin, all minors are positive!

k	2	3	4	5	6	7	8	9	10
ℓ_c	1	2	2	3	3	4	4	5	5

This positivity exploits the hierarchy of minors.

The EFTHedron in the real world

The geometry is richer in the real world

$$\begin{array}{cccccc}
 & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\
t^0 & g_{0,0} & g_{1,0} & g_{2,0} & g_{3,0} & \cdots \\
t^1 & & g_{0,1} & g_{1,1} & g_{2,1} & \cdots \\
t^2 & & & g_{0,2} & g_{1,2} & \cdots \\
t^3 & & & & g_{0,3} & \cdots
\end{array}$$

- For fixed mass-dimension, there is a critical spin above which it becomes cylic (all ordered minors are positive)
- The boundaries are determined from the cylicity

$$\langle X, i, i+1\rangle > 0 \text{ for}, \quad i \geq 5, \langle X, 4, 3\rangle > 0, \quad \langle X, 3, 5\rangle > 0$$

The EFTHedron in the real world

The geometry is richer in the real world

$$\begin{array}{cccccc} & m^0 & \frac{1}{m^2} & \frac{1}{m^4} & \frac{1}{m^6} & \cdots \\ t^0 & g_{0,0} & \color{red}{g_{1,0}} & \color{red}{g_{2,0}} & \color{red}{g_{3,0}} & \cdots \\ t^1 & & \color{red}{g_{0,1}} & \color{red}{g_{1,1}} & \color{red}{g_{2,1}} & \cdots \\ t^2 & & & g_{0,2} & g_{1,2} & \cdots \\ t^3 & & & & g_{0,3} & \cdots \end{array}$$

- The boundaries of the Minkowski sum is always given by that of the highest k

$$\partial\left[\begin{pmatrix} g_{1,0} \\ g_{0,1} \end{pmatrix} \oplus \begin{pmatrix} g_{2,0} \\ g_{1,1} \end{pmatrix} \oplus \begin{pmatrix} g_{3,0} \\ g_{2,1} \end{pmatrix}\right] = \partial\begin{pmatrix} g_{3,0} \\ g_{2,1} \end{pmatrix}$$

- The moment curve constraint is generalized to rescaled moment curves

$$\sum_i (1, x_i, x_i^2, \cdots) \to \sum_i (1, x_i, \color{red}{\gamma} x_i^2, \cdots)$$

The EFTHedron in the real world

The same structure is found for when the external states are massless with spins: photons, gauge bosons, and gravitons:

$$G^{\alpha}_{\ell_i}(\cos\theta) \rightarrow d^{\ell}_{h_1-h_2,\,h_3-h_4}(\theta) = \langle \ell, h_1 - h_2 | e^{-i\theta\mathcal{J}_y} | \ell, h_3 - h_4 \rangle$$

We simply replace Gegenbauer polynomials with Wigner d-function. For $(-h, h, h, -h)$ we simply have

$$d^{\ell}_{-2h,2h}(\theta) = \mathcal{J}(\ell + 4h, 0, -4h, \cos\theta)$$

The EFTHedron in the real world

Consider the configuration (-2, +2, +2, -2) where we have

$$\langle 14\rangle^4[23]^4\left(\sum_{i,j} g_{i,j} z^i t^j\right) \tag{8}$$

The exchanged spin begins with spin-4

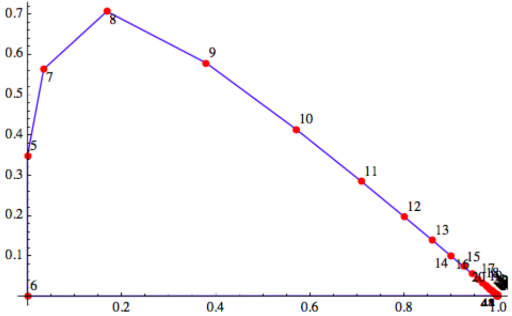
- (z^2,t^2): The space is one-dimensional, and the bound is simply

$$-\frac{11}{36} < \frac{g_{2,0}}{g_{0,2}}$$

- (z^4,z^2t^2,t^4): The critical spin is $s_c=6$, spin-4 is inside the hull, i.e. not a vertex. The boundaries are:

$$\langle X,i,i+1\rangle > 0 \text{ for}, \quad i\geq 7, \langle X,6,5\rangle>0, \quad \langle X,5,7\rangle>0 \tag{9}$$

Out[175]=



(10)

The space of CFT from $\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle$

Consider the a 1D four-point function:

$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle \equiv F(z)$$

$$\mathbf{F}(z) = \sum_{\Delta} \mathfrak{p}_{\Delta} C_{\Delta}(z), \quad C_{\Delta}(z) = z^{\Delta}\, {}_2F_1(\Delta, \Delta, 2\Delta, z)$$

Expand the four-point function, around $z = \frac{1}{2}$

$$\mathbf{F}\left(\frac{1}{2} + y\right) = \sum_{q=0}^{\infty} f_q y^q$$

We consider the space $\{f_q\}$

Crossing symmetry

$$z^{-2\Delta_\phi} F(z) = (1-z)^{-2\Delta_\phi} F(1-z) \to F(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi} F(1-z)$$

implies the four-point function lies in a subplane X

The space of CFT from $\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle$

The 1-D blocks also yield an infinite set of vectors

$$C_\Delta\left(\frac{1}{2}+y\right)=\sum_{q=0}^{\infty}c_{\Delta,q}y^q$$

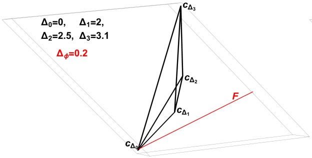
Unitarity then requires that

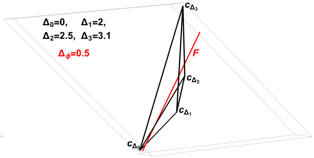
$$\mathbf{F}=\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{pmatrix}\in\sum_\Delta \mathfrak{p}_\Delta \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix} \quad \mathfrak{p}_\Delta>0$$

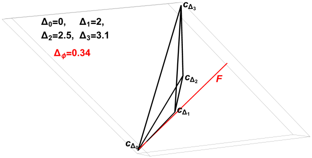
The space of CFT from $\langle\phi(1)\phi(2)\phi(3)\phi(4)\rangle$

$$\mathbf{F} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{pmatrix} \in \sum_{\Delta} \mathfrak{p}_\Delta \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix} \qquad \mathfrak{p}_\Delta > 0$$

For a given CFT spectrum have the polytope $P(\Delta_i) = \sum_i \mathfrak{p}_{\Delta_i} \vec{c}_{\Delta_i}$ and a crossing plane $X(\Delta_\phi)$, and they must intersect. For example:







The CFTHedron

Is there a similar structure?

Indeed there is!

$$Det \begin{pmatrix} C_{\Delta_1}(z_1) & C_{\Delta_2}(z_1) & \cdots & C_{\Delta_n}(z_1) \\ C_{\Delta_1}(z_2) & C_{\Delta_2}(z_2) & \cdots & C_{\Delta_n}(z_2) \\ \vdots & \vdots & \vdots & \vdots \\ C_{\Delta_1}(z_n) & C_{\Delta_2}(z_n) & \cdots & C_{\Delta_n}(z_n) \end{pmatrix} >$$

for $z_1 < z_2 < \cdots < z_n$ and $\Delta_1 < \Delta_2 < \cdots < \Delta_n$

The convex hull of the block vectors is again a cyclic polytope!

The CFTHedron

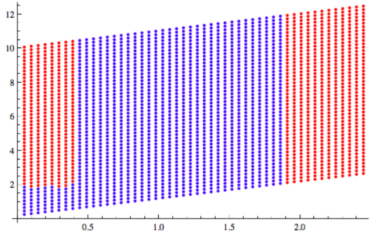
This gives us the control over the relevant boundaries

$$\mathbf{F} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{pmatrix} \in \sum_{\Delta} \mathfrak{p}_{\Delta} \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix} \qquad \mathfrak{p}_{\Delta} > 0$$

For example with with $D = 1$, (f_0, f_2) the relevant boundaries are

$$\begin{aligned} W_1 &= (1\Delta_1\Delta_2)\,, \\ W_2 &= (\infty 1\Delta_1)\,, \\ W_3 &= (\infty\Delta_1\Delta_2)\,, \\ W_4 &= (1\Delta_2\dot{\Delta}_2)\,, \\ W_5 &= (\infty\Delta_2\dot{\Delta}_2)\,. \end{aligned}$$

The resulting carved out space is

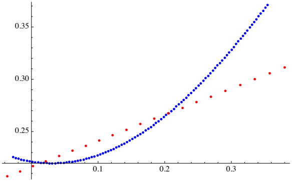


The CFTHedron

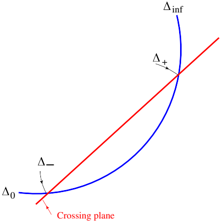
This can be simply understood as considering the condition:

$$\langle Id, F, \Delta_i, \Delta_{i+1} \rangle > 0$$

Projecting through $Id = (1, 0, 0, 0)$, we have a two-dimension geometry



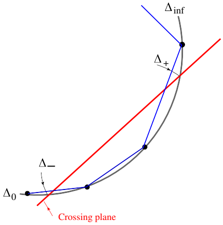
$, \quad \rightarrow$

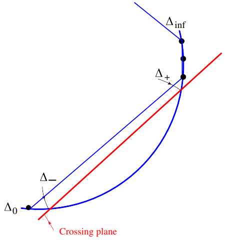


where $\Delta_\pm \rightarrow \langle F, 1, \Delta \rangle = 0$

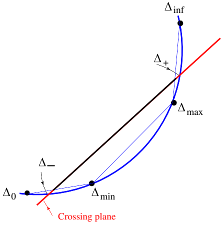
The CFTHedron

Constraint on the specturm

allowed : 

, not allowed : 

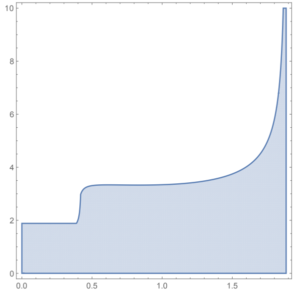
As well as the four-point function



The CFTHedron

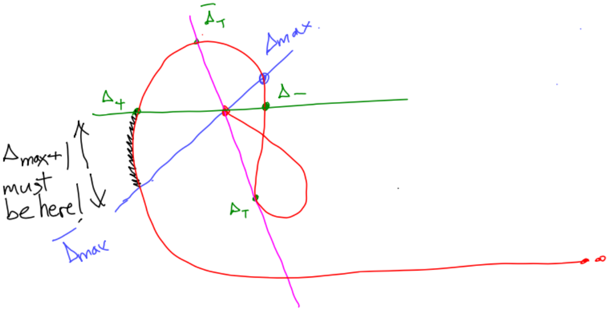
At the next order we have (f_0, f_2, f_4) inside a four-dimensional polytope.

Exp, given $\Delta_\phi = 0.3$, in the space of possible lowest first two operators (Δ_1, Δ_2) are given by:



The CFTHedron

We can also understand this plot from the geometry. Projecting through *Id* and *X*:



The task is that the spectrum must form a triangle that contains the origin.

Conclusions

The constraint of unitarity, locality and symmetries manifest itself as positive geometry on the space of consistent QFTs.

For EFTs the space of consistent coupling constant *is* the EFThedron

For CFTs the space is given by the combinatorics arising from the intersection of the crossing plane with the cyclic polytope

- For the $s-u$ EFThedron, the space for generalized moment curve remains to be explored.
- For practical bounds, explore the space for mixed graviton photon scattering
- Proof of various conjectures (Weak gravity) for the land scape.
- Solving the 1D CFT geometry at higher dimensions (in external data)
- Extensions to CFT with $D > 1$ expose the CFThedron