The Vacua of Some 2+1 Dimensional Gauge Theories

Zohar Komargodski

Simons Center for Geometry and Physics

June 25, 2018

We'll explain some recent non-perturbative results about gauge theories in 3 (2+1) dimensions. We will then see that there are also interesting implications for gauge theories in 4 dimensions.

The presentation is mostly based on collaborations with Davide Gaiotto, Jaume Gomis, Nathan Seiberg, as well as some work in progress.

Different phases of Quantum Field Theory that we will encounter today:

- **Trivial Gapped**: No massless excitations, no topological theory, trivial (product) wave function
- **Topological Field Theory (TFT)**: no massless excitations, but some long range entanglement and topological order (such as anyons). Nontrivial ground state wave function.
- **Massless phases**: This could be due to a Conformal Field Theory, or due to Nambu-Goldstone particles.

We will concentrate our attention on two models:

- $SU(N)$ adjoint *Majorana* fermion λ_α coupled to $SU(N)$ gauge fields with a Chern-Simons term at level k. We refer to this theory as "Adjoint QCD_3."
- N_f $SU(N)$ fundamental *complex* fermions λ_α coupled to $SU(N)$ gauge fields with a Chern-Simons term at level k. We refer to this theory simply as "QCD_3."

These models have various weakly coupled limits where we can understand the dynamics in detail. There are also strongly coupled regimes, where we have made some conjectures.

Typically, 't Hooft anomaly matching conditions (along with many other constraints) are an important ingredient in our study of strongly coupled theories. One may be worried that since we are studying 3 dimensional theories, this important tool will not be available to us.

However, recent developments uncovered a swath of new discrete anomalies.

- If a [fermionic] QFT in 2+1 dimensions has a time reversal symmetry with $T^2 = (-1)^F$, there is a possible obstruction to studying this theory on a non-orientable space. The obstruction is valued in $\mathbb{Z}_{16}$ (see [Witten 1508.04715] for an exposition).
- Mixed 't Hooft anomalies between time reversal symmetry and global symmetries. 't Hooft anomalies for global symmetries.
- If there is a $\mathbb{Z}_N$ one-form symmetry, there is an anomaly valued in $\mathbb{Z}_N$ (for fermionic theories).

Matching these discrete anomalies is, for practical purposes, as constraining and non-trivial as matching continuous 't Hooft anomalies.

The challenge is often, however, in computing these anomalies. Even in rather simple theories, the computation of these anomalies may not be easy. We will see examples.

The Lagrangian is

$$\mathcal{L} = \frac{-1}{4g^2}\mathrm{Tr}F^2 + \frac{k}{4\pi}\mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) + \mathcal{L}_{matter} \ .$$

- Adjoint QCD: $\mathcal{L}_{matter} = i\bar{\lambda}\not{D}\lambda + \frac{m}{4\pi}\bar{\lambda}\lambda$
- QCD: $\mathcal{L}_{matter} = \sum_{k=1}^{N_f}\left(i\bar{\psi}^k\not{D}\psi_k + \frac{m}{4\pi}\bar{\psi}^k\psi_k\right)$

Note that m is a real parameter.

Consistency requires

$$\text{Adjoint QCD}: \quad \frac{N}{2} + k \in \mathbb{Z}\ .$$

$$\text{QCD}: \quad \frac{N_f}{2} + k \in \mathbb{Z}\ .$$

Adjoint QCD has no ordinary global symmetries while QCD has $U(N_f)$ symmetry. Note that m preserves the $U(N_f)$ symmetry!

In both cases, if $k = 0$ and $m = 0$, we have time reversal symmetry and $T^2 = (-1)^F$ (cf. Seiberg's talk).

While Adjoint QCD has no ordinary (zero-form) global symmetries, it has a $\mathbb{Z}_N$ one-form symmetry because the center does not act on the matter fields. (We use the terminology of [Kapustin-Seiberg], [Gaiotto-Kapustin-Seiberg-Willett].)

Decoupling Limits: $|m| \gg g^2$ or $k \gg N$.

In some corners of the parameter space the theory becomes weakly coupled.

If $|m| \gg g^2$ the quarks decouple even before the interactions set in. One has to be careful integrating them out as there is a non-decoupling effect [Redlich, Niemi-Semenoff] proportional to $m/|m| = sgn(m)$.
This shifts k according to

$$\text{Adjoint QCD} : k \to k + sgn(m)\frac{N}{2} \ .$$

$$\text{QCD} : k \to k + sgn(m)\frac{N_f}{2} \ .$$

Another weak coupling limit is

$$k \gg N$$

The gauge field A now has a mass kg^2 and therefore it decouples before the interactions set in:

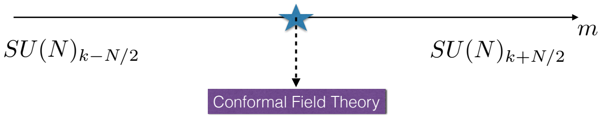
$$kg^2 \gg g^2 N \ .$$

The remaining light fields are weakly interacting and there is a weakly coupled *Conformal Field Theory* if we tune m.

Therefore the physics of large m for any k is a (possibly trivial) TFT. For large k there is a second order phase transition as we vary m, described by some (weakly coupled) Chern-Simons matter theory. On the two sides of the transition we encounter different TFTs so these transitions are typically non-Landau-Ginzburg.

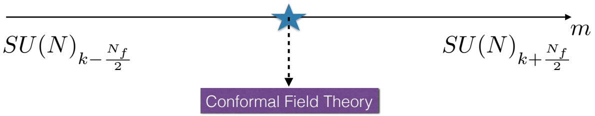
We therefore suggest the following phase diagram for $k \gg N$ ("large k") in Adjoint QCD:

$$SU(N) + \lambda_\alpha, \text{ level } k \qquad\qquad k \gg N$$



And an analogous phase diagram in QCD for $k \gg 1$:

$$SU(N) + N_f \psi, \quad \text{level } k \qquad\qquad k \gg 1$$



The interesting question is where does this picture break down. This is analogous to asking where is the end of the conformal window. Surprisingly, in 3d there is a very concrete proposal for the answer!

Adjoint QCD: $k \geq N/2$

QCD: $k \geq N_f/2$

These bounds are motivated by the consistency of the whole picture we will shortly propose. Some preliminary observations:

- Adjoint QCD has $\mathcal{N}=1$ supersymmetry for $m=-kg^2$. It is known [Witten] to be unbroken for $k \geq N/2$ and the Witten index precisely agrees with the number of lines in $SU(N)_{k-N/2}$ TFT.
- QCD has a dual bosonic description in terms of $U\left(k+\frac{N_f}{2}\right)$ gauge theory coupled to N_f fundamental bosons [...Aharony et al., Minwalla et al....]. This duality needs to be fixed for $k < N_f/2$ because the bosonic theory has massless Goldstone modes in the Higgs phase.

To describe what happens for $k < N/2$ and $k < N_f/2$ one has to make a certain leap, as this is a strongly coupled regime. We of course know what happens at large $|m|$, where we can just integrate out the quarks and flow to a Chern-Simons TFT, as above. The question is what happens when the matter fields are light.

It is useful to start from the most strongly coupled regime, where $k = 0$. We have time reversal symmetry at $m = 0$ and an associated anomaly.

Let us start with the conjecture for Adjoint QCD:

$SU(N)$+Adjoint, $m = k = 0 \longrightarrow\longrightarrow U\left(\frac{N}{2}\right)_{\frac{N}{2},N} \otimes G_\alpha$, where G_α is a massless Majorana fermion.

The $m = k = 0$ theory has "accidental" $\mathcal{N} = 1$ supersymmetry and the G_α can be interpreted as the Goldstino particle. Adding some small δm in the UV would lift this particle but the TFT $U\left(\frac{N}{2}\right)_{\frac{N}{2},N}$ would remain. Therefore, this new quantum phase exists in a neighbourhood of $m = 0$ and there are certain phase transitions to the semi-classical domain.

The conjecture that the infrared theory contains $U\left(\frac{N}{2}\right)_{\frac{N}{2},N}$ miraculously saturates all the anomalies:

- It has $\mathbb{Z}_N$ one-form symmetry and it can be checked that it has exactly the right one-form symmetry anomaly.
- Time Reversal Symmetry: From level-rank duality,

$$U\left(\frac{N}{2}-k\right)_{\frac{N}{2}+k,N} \simeq U\left(\frac{N}{2}+k\right)_{-\frac{N}{2}+k,-N} ,$$

we see that $U\left(\frac{N}{2}\right)_{\frac{N}{2},N}$ is time-reversal invariant if $k=0$, as required!! Furthermore it has a time reversal anomaly [...Tachikawa-Yonekura,Cheng...]

$$\nu = 2(-1)^{N/2+1} .$$

From the Goldstino G_α we get at $\nu(Goldstino) = 1$. So the total infrared time reversal anomaly at $k = m = 0$

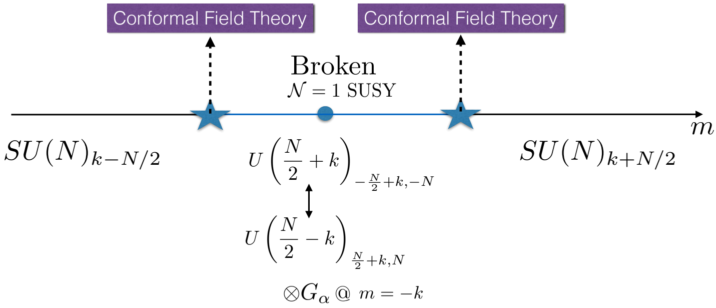
$$v_{IR} = 1 + 2(-1)^{N/2+1} \mod 16 = \begin{cases} 3 \mod 16 & \text{if} \quad N = 2 \mod 4 \\ -1 \mod 16 & \text{if} \quad N = 0 \mod 4 \end{cases}$$

This is in agreement with the ultraviolet anomaly because it is the same as $N^2 - 1 \mod 16$.

It may appear surprising that the massless $k = 0$ theory, which is essentially the 2+1 dimensional analog of 3+1 dimensional adjoint QCD, is not confining. Indeed, the Wilson lines have a perimeter law and a fusion/braiding algebra that is dictated by the TFT in the quantum phase.

We therefore propose the following phase diagram for $0 \leq k < N/2$:

$$SU(N) + \lambda_\alpha, \text{ level } k \qquad 0 \leq k < N/2$$



The transitions from the "quantum" phase to the semi-classical phases have a new dual description, for example, the left transition could be described with

$$U\left(\frac{N}{2}-k\right)+\hat{\lambda}$$

with levels $\frac{3}{4}N+\frac{k}{2}, N$. $\hat{\lambda}$ is a dual fermion in the adjoint representation.

This is a new adjoint-adjoint Fermion-Fermion duality.

The phase diagram for QCD for $0 \leq k < N_f/2$ is constructed using an analogous procedure. For even (sufficiently small) N_f and $m = k = 0$ we make the following conjecture:

$SU(N)+N_f\ \psi$, $m = k = 0 \longrightarrow\longrightarrow \mathcal{M} = \frac{U(N_f)}{U\left(\frac{N_f}{2}\right)\times U\left(\frac{N_f}{2}\right)}$ and the sigma model is accompanied by a Wess-Zumino term, Γ, whose coefficient is N.

In short, the symmetry is broken as

$$U(N_f) \to U\left(\frac{N_f}{2}\right) \times U\left(\frac{N_f}{2}\right)$$

The UV theory has time reversal symmetry anomaly $\nu_{UV} = 2N_f N \mod 16$. It is not trivial to compute the time reversal anomaly of the infrared sigma model. The Wess-Zumino term Γ is crucial in order to get the right answer.

In the particular case of $N_f = 2$, this is the $\mathbb{C}P^1$ model at $\theta = \pi$ discussed in [Freed,ZK,Seiberg]. In work in progress [Hason,ZK,Thorngren] we show that the anomaly in fact matches. In addition, we show that all the discrete flavour symmetry anomalies match.

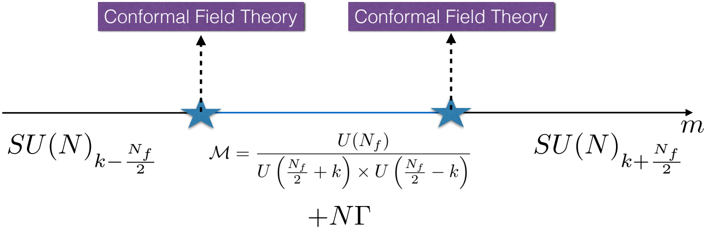
In general, we propose the following symmetry breaking pattern

$$U(N_f) \to U\left(\frac{N_f}{2} + k\right) \times U\left(\frac{N_f}{2} - k\right)$$

(and the sigma model again needs to be accompanied by a Wess-Zumino term).

This makes sense only for $k < N_f/2$ which is another way to understand the boundary between this phase and the "large k" phase. (Note that it is consistent with the [Vafa-Witten] theorem.)

$$SU(N) + N_f\psi, \quad \text{level } k \qquad\qquad 0 \leq k < \frac{N_f}{2}$$



As in the previous case, the transition has a new dual description. For instance, for the left transition it is given by

$$U\left(\frac{N_f}{2}-k\right)_N + N_f\phi \ .$$

While the quantum phases are robust, the transitions in these various cases do not actually need to be 2nd order.

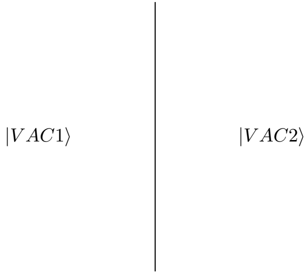
See [...Jensen-Karch, Armoni-Niarchos, Argurio-Bertolini-Bigazzi-Cotrone-Niro...] for stringy constructions of these dualities and symmetry breaking phases.

See also [Karthik-Narayanan] for a recent lattice study, (strongly) hinting that symmetry breaking indeed takes place!

We will now mention (very) briefly a striking parallel between the problem of $3d$ dynamics that we have been analyzing so far and the problem of $4d$ dynamics and domain walls.

- 4d Yang-Mills theory with an Adjoint fermion.
- 4d Yang-Mills theory with fundamental Dirac fermions (a.k.a QCD).

In both cases, under appropriate circumstances, there are degenerate vacua, each of which is gapped and trivial. Therefore, at low energies there are some 3d theories living on these domain walls.



These domain walls have the same symmetries and anomalies as the pure 3d theories. In some cases their phases can be explicitly computed using 4d techniques and one can discover the quantum phases that have appeared above.

For instance, 4d QCD with $N_f = 2$ massive quarks and $\theta = \pi$ has two ground states for any nonzero positive mass. For large m we have $SU(2)_1$ TFT on the wall and for small m we find the coset $\frac{U(2)}{U(1)\times U(1)} \simeq \mathbb{CP}^1$ on the wall (while the bulk is massive!).

Conversely, our 3d techniques can shed light on some open problems in 4d dynamics. We will now make a proposal for the domain walls theories of $\mathcal{N} = 1$ four-dimensional gauge theories with simply connected gauge groups. We identify the domain wall theory with the quantum phase in the corresponding 3d adjoint QCD model. This problem has been resistant to other approaches for quite some time.

- $\mathcal{N}=1$ Super Yang-Mills with gauge group $SU(N)$ [Agrees with Acharya-Vafa]:

$$U(i)_{N-i,N} \ .$$

- $\mathcal{N}=1$ Super Yang-Mills with gauge group $Sp(N)$:

$$Sp(i)_{N-i+1} \ .$$

- $\mathcal{N}=1$ Super Yang-Mills with gauge group $Spin(N)$ [Gomis-Seiberg-ZK, Cordova-Hsin-Seiberg]:

$$O\left(i\right)^1_{N-i-2,N-i+1} \ .$$

We have seen that discrete anomalies are a powerful constraint on the dynamics of QFTs. For additional discussions, see, for instance, the recent papers [**Benini** et al., **Cordova** et al., **Tanizaki**, **Sulejmanpasic**, **Kikuchi**, **Misumi**, **Sakai**, **Shimizu**, **Anber-Poppitz**, **Aitken-Cherman-Ünsal**, **Yao-Hsieh-Oshikawa**, **Dunne**, **Hofman-Iqbal**,**Guo-Putrov-Wang**, **Yamazaki-Yonekura**, **Kitano** et al., **Di Vecchia** et al., ...]

It has been further argued that such discrete anomalies are, in special cases, the continuum avatar of Lieb-Schultz-Mattis theorems in lattice theories [ZK-**Sulejmanpasic-Ünsal**, **Metlitski-Thorngren**, ...]

We have briefly seen that domain walls in four-dimensional (supersymmetric and non-supersymmetric) theories exhibit rich dynamics. Some additional aspects are discussed in [... **Dierigl-Pritzel**, **Draper**, **Ritz-Shukla**, **Armoni-Niarchos**, **Argurio-Bertolini-Bigazzi-Cotrone-Niro** (embedding in the Sakai-Sugimoto model – The bosonic dual appears naturally!), **Aitken-Baumgartner-Karch** (interactions between domain walls and the corresponding 3d quivers and dualities; *to appear*)].

Thank you for the attention !!