Quantum Entanglement in Holography

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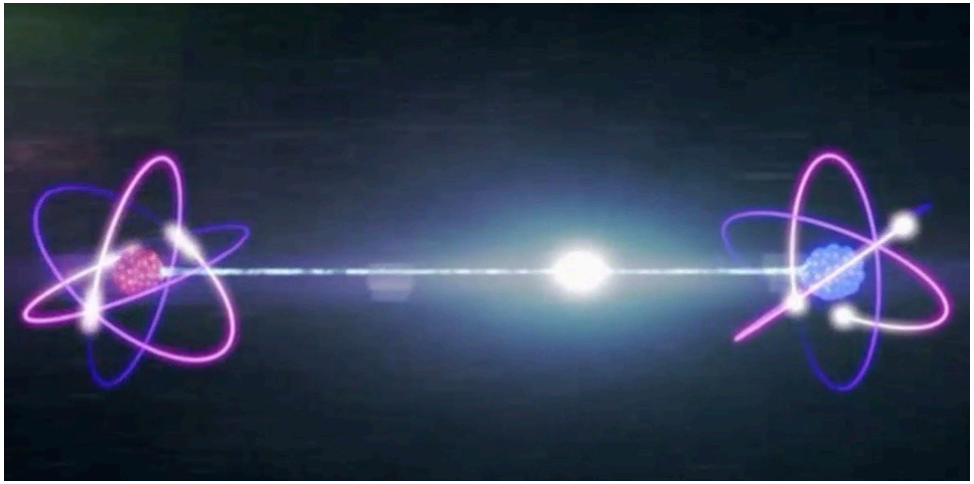
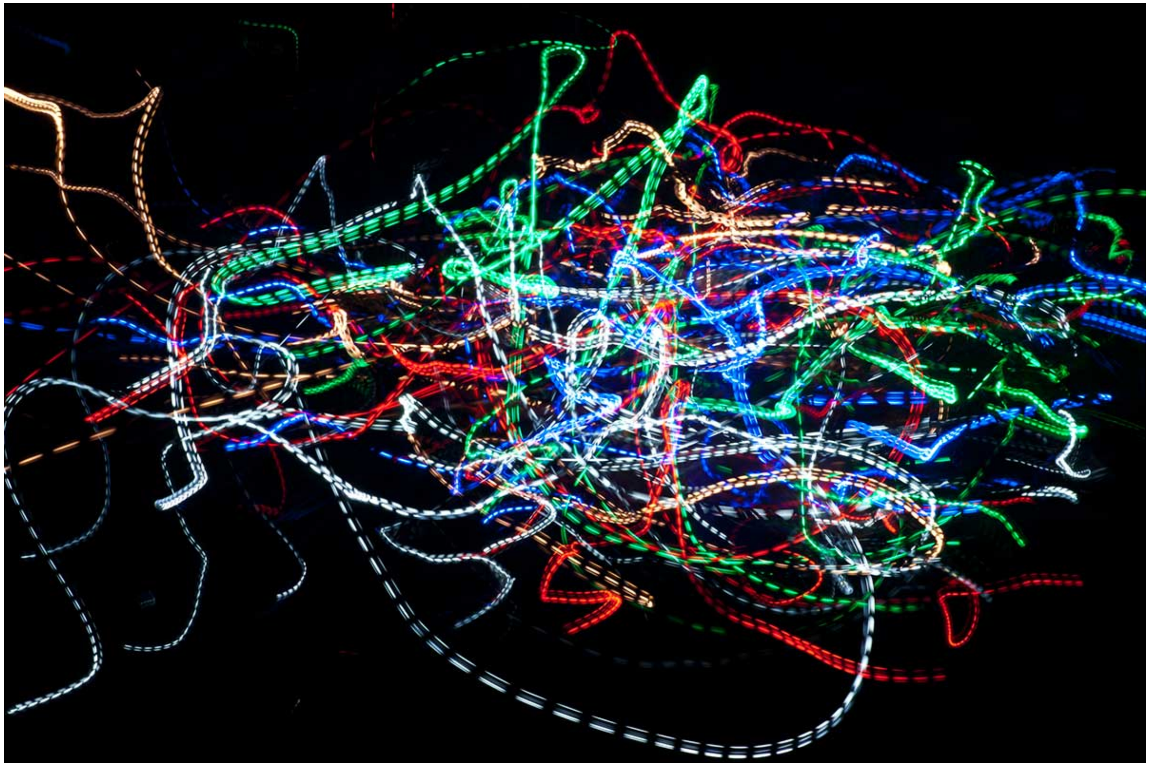
Quantum entanglement

- Correlation between parts of the full system
- Simplest example: $|\Psi\rangle = |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$



Quantum entanglement

General case:



Entanglement is complicated

- Entanglement is not just entanglement entropy.
- “Structure” of (many-body) quantum state.
- Entanglement entropy is one of many probes.
- Other probes: correlators, Renyi entropy, modular Hamiltonian/flow, complexity, quantum error correction, …

Entanglement is complicated

- Structure of entanglement is increasingly important in the study of many-body physics.
- String theory and quantum field theory are (special) many-body systems.
- Strategy: study patterns of entanglement to address interesting questions such as:

➢ How does quantum gravity work?

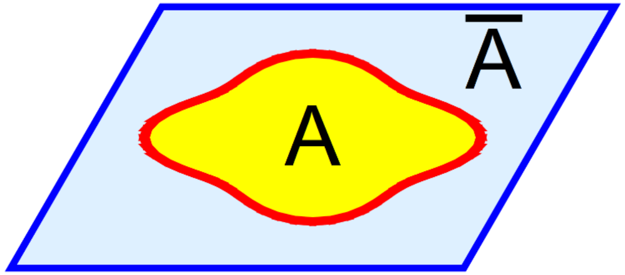
➢ How do we describe the black hole interior?

➢ How do we describe cosmology?

Plan

- Entanglement in QFT
- Entanglement in holography
- Applications and generalizations
- Holographic error correction

Entanglement entropy in QFT

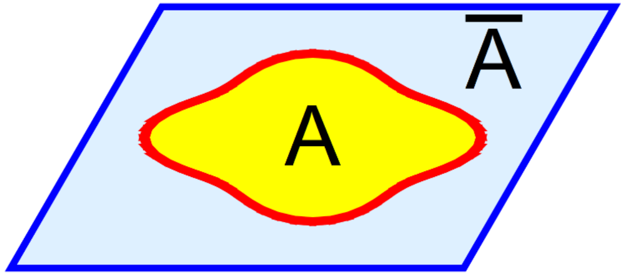


- Von Neumann entropy: $S \stackrel{\text{def}}{=} -\mathrm{Tr}\,(\rho_A \ln \rho_A)$

- Renyi entropy: $S_n \stackrel{\text{def}}{=} \frac{1}{1-n} \ln \mathrm{Tr}\,\rho_A^n$

- Interesting in QFT and holography

Entanglement entropy in QFT



- Area-law divergence: $S = \# \frac{\text{Area}}{\epsilon^{d-2}} + \cdots + S_{univ} + \cdots$
- Even d: $S_{univ} = \left(\sum_i c_i I_i\right) \ln \epsilon$

➢E.g. CFT_4: $S_{univ} = \left(\int aR + aK^2 + cW\right) \ln \epsilon$

- Odd d : S_{univ} is finite and nonlocal.

➢E.g. spherical disk: $S_{univ} \sim F$

- Can generalize to Renyi entropy
- Can generalize to cases with corners

[Allais, Balakrishnan, Banerjee, Bhattacharya, Bianchi, Bueno, Carmi, Chapman, Czech, XD, Dowker, Dutta, Elvang, Faulkner, Fonda, Galante, Hadjiantonis, Hubeny, Klebanov, Lamprou, Lee, Leigh, Lewkowycz, McCandish, McGough, Meineri, Mezei, Miao, Myers, Nishioka, Nozaki, Numasawa, Parrikar, Perlmutter, Prudenziai, Pufu, Rangamani, Rosenhaus, Safdi, Seminara, Smolkin, Solodukhin, Sully, Takayanagi, Tonni, Witczak-Krempa, …]

Strong subadditivity (SSA)

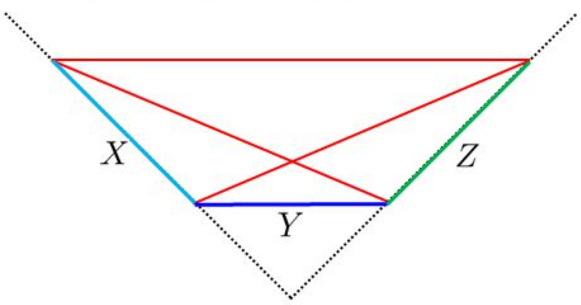
$$S_{AB} + S_{BC} \geq S_{ABC} + S_A$$

- In Lorentz-invariant QFT, leads to:

➢Monotonic C-function in 2d and F-function in 3d

➢A-theorem in 4d

- Basic idea in 2d:



- A particular 2nd derivative of $S(l)$ is non-positive.
- Define $C(l) = 3l\, S'(l) \quad \Rightarrow \quad C'(l) \leq 0$

[Casini, Huerta, Teste, Torroba]

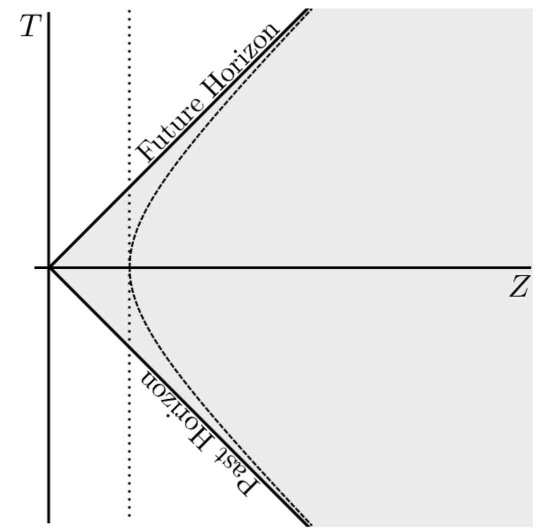
Modular Hamiltonian

$$K \stackrel{\text{def}}{=} -\ln \rho$$

- Makes the state look thermal
- Nonlocal in general
- Exceptions:

1. Thermal state $\rho = e^{-\beta H}/Z$
2. Half space in QFT vacuum:

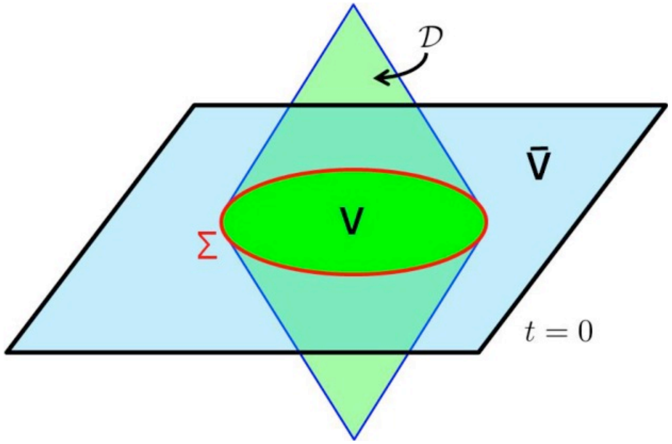
$$K \sim 2\pi \int z\, T^{00}$$



[Bisognano, Wichmann]

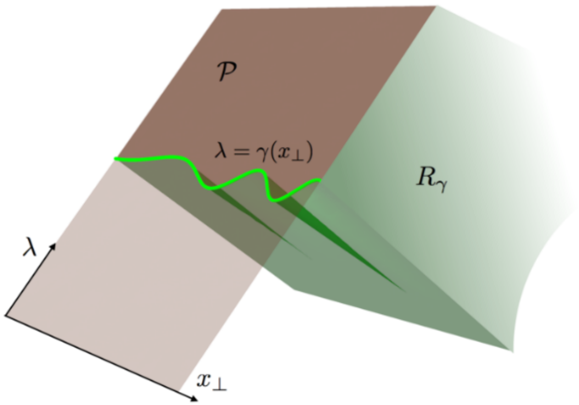
Modular Hamiltonian

3. Spherical disk in CFT vacuum



$$K \sim 2\pi \int \frac{R^2 - r^2}{R} T^{00}$$

- Generalizes to wiggly cases



[Casini, Huerta, Myers, Teste, Torroba]

Modular Hamiltonian

- Shape deformation governed by $T_{\mu\nu}$, leading to:

➢ Averaged Null Energy Condition (ANEC):

$$\int dx^+ \langle T_{++} \rangle \geq 0$$

- Uses monotonicity of relative entropy (⇔ SSA)
- Relative entropy $S(\rho|\sigma) \stackrel{\text{def}}{=} \mathrm{Tr}(\rho \ln \rho) - \mathrm{Tr}(\rho \ln \sigma)$
- $S(\rho|\sigma) \geq 0$; $\quad S(\rho|\sigma) = 0$ iff $\rho = \sigma$
- A measure of distinguishability
- Monotonicity: $S(\rho_{AB}|\sigma_{AB}) \geq S(\rho_A|\sigma_A)$

[Faulkner, Leigh, Parrikar, Wang; Hartman, Kundu, Tajdini; Bousso, Fisher, Leichenauer, Wall; Balakrishnan, Faulkner, Khandker, Wang]

Modular Hamiltonian

- Shape deformation governed by $T_{\mu\nu}$, leading to:

➢ Quantum Null Energy Condition (QNEC):

$$\langle T_{++} \rangle \geq \frac{1}{2\pi} S''$$

- Uses causality (in modular time)

[Faulkner, Leigh, Parrikar, Wang; Hartman, Kundu, Tajdini; Bousso, Fisher, Leichenauer, Wall; Balakrishnan, Faulkner, Khandker, Wang]

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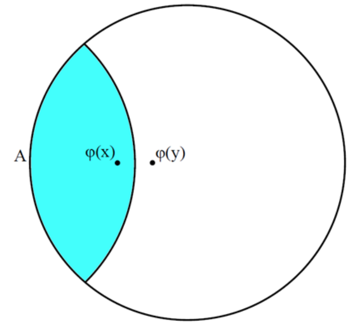
- Entanglement in QFT
- **Entanglement in holography**
- Applications and generalizations
- Holographic error correction

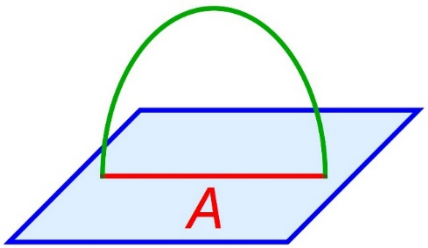
Holographic Entanglement Entropy

In holography, von Neumann entropy is given by the area of a dual surface:

[Ryu & Takayanagi '06]

$$S = \min \frac{\text{Area}}{4G_N}$$





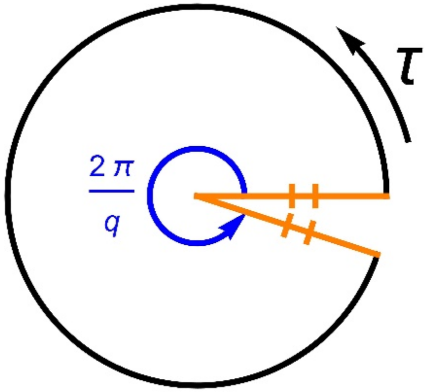
- Practically useful for understanding entanglement in strongly-coupled systems.

[Huijse, Sachdev & Swingle '11]
[XD, Harrison, Kachru, Torroba & Wang '12; …]

- Conceptually important for understanding the emergence of spacetime from entanglement.

[Van Raamsdonk '10; Maldacena & Susskind '13; …]

Proof from AdS/CFT



Basic idea:

- Use replica trick to calculate Renyi entropy S_n.
- Given by the action of a bulk solution B_n with a conical defect.
- Von Neumann entropy: $S = \partial_n I[B_n]|_{n=1}$.
- The RT minimal surface is a relic of the conical defect as deficit angle $2\pi\left(1 - \frac{1}{n}\right)$ goes to zero.
- Variation of on-shell action is a boundary term, giving the area.

[Lewkowycz, Maldacena]

HRT in time-dependent states

- Strictly speaking, the RT formula applies at a moment of time-reflection symmetry.

- For time-dependent states, we have a covariant generalization (HRT): [Hubeny, Rangamani & Takayanagi '07]

$$S = \text{ext}\,\frac{\text{Area}}{4G_N}$$

- Powerful tool for studying time-dependent physics such as quantum quenches.

- Can also be derived from AdS/CFT. [XD, Lewkowycz & Rangamani '16]

Corrections to RT formula

The RT formula has been refined by:

- Higher derivative corrections

- Quantum corrections

Higher derivative corrections to RT

- Higher derivative corrections (α'):

$$S = \text{ext}\frac{A_{\text{gen}}}{4G_N}$$

[XD '13; Camps '13; XD & Lewkowycz, 1705.08453; ...]

- "Generalized area" $A_{\text{gen}} = \partial_n I[B_n]|_{n=1}$
- Has the form: $S_{\text{Wald}} + S_{\text{extrinsic curvature}}$
- Example:

$$L = -\frac{1}{16\pi G_N}\left(R + \lambda R_{\mu\nu}R^{\mu\nu}\right) \xrightarrow{\text{yields}} A_{\text{gen}} = \int_X 1 + \lambda\left(R_a^a - \frac{1}{2}K_a K^a\right)$$

- Applies to (dynamical) black holes and shown to obey the Second Law.

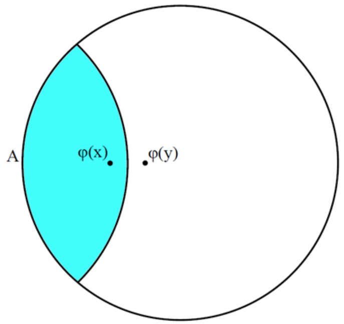
[Bhattacharjee, Sarkar & Wall '15; Wall '15]

Quantum corrections to RT

These corrections ($G_N \sim 1/N^2$) come from matter fields and gravitons.

- The prescription is surprisingly simple:

$$S = \text{ext}\left(\frac{\langle A \rangle}{4G_N} + S_{\text{bulk}}\right)$$



[Engelhardt & Wall '14; XD & Lewkowycz, 1705.08453]

- Quantum extremal surface.
- Valid to all orders in G_N.
- Natural: invariant under bulk RG flow.
- Matches one-loop FLM result. [Faulkner, Lewkowycz & Maldacena '13]
- S_{bulk} defined in entanglement wedge (domain of dependence of achronal surface between A and γ_A)

Quantum corrections to RT

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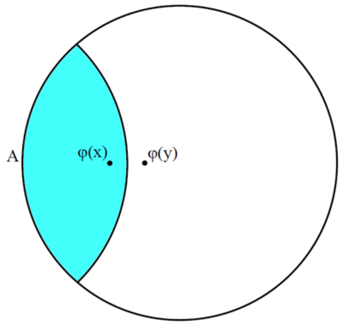


- Can be derived from AdS/CFT. [XD & Lewkowycz 1705.08453]
- Example: 2d dilaton gravity with N_f matter fields. [CGHS; Russo, Susskind & Thorlacius '92]

➢ Quantum effects generate nonlocal effective action:

$$L = -\frac{1}{2\pi}\left[e^{-2\phi}(R + 4\lambda^2) + \frac{N_f}{96\pi} R \frac{1}{\nabla^2} R\right]$$

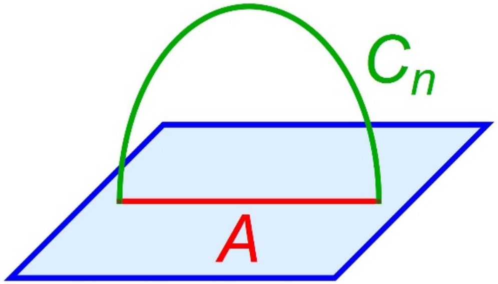
➢ Appears local in conformal gauge. Can check quantum extremality.

Holographic Renyi entropy

$$S_n \stackrel{\text{def}}{=} \frac{1}{1-n} \ln \mathrm{Tr} \rho_A^n$$

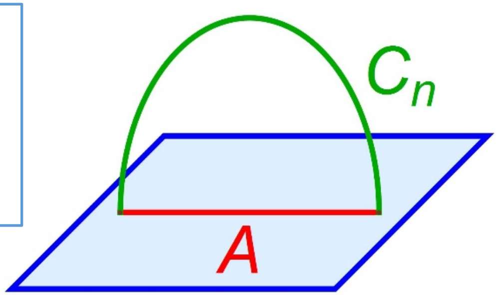
- Given by cosmic branes instead of minimal surface

$$n^2 \partial_n \left(\frac{n-1}{n} S_n \right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4 G_N}$$



[XD]

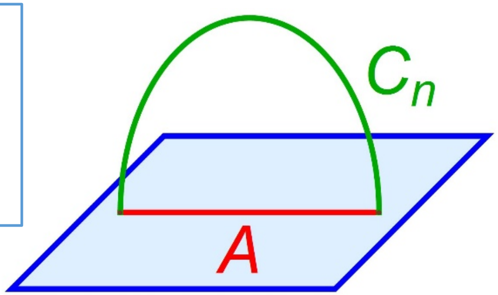
$$n^2 \partial_n \left(\frac{n-1}{n} S_n\right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$



- Cosmic brane similar to minimal surface; they are both codimension-2 and anchored at edge of A.
- But brane is different in having tension $T_n = \frac{n-1}{4nG_N}$.
- Backreacts on ambient geometry by creating conical deficit angle $2\pi \frac{n-1}{n}$.
- Useful way of getting the geometry: find solution to classical action $I_{\text{total}} = I_{\text{bulk}} + I_{\text{brane}}$.
- As $n \to 1$: probe brane settles at minimal surface.
- One-parameter generation of RT.

[XD]

$$n^2 \partial_n \left(\frac{n-1}{n} S_n\right) = \frac{\text{Area}(\text{Cosmic Brane}_n)}{4G_N}$$



- Why does this area law work?
- Because LHS is a more natural candidate for generalizing von Neumann entropy:

$$\widetilde{S_n} \overset{\text{def}}{=} n^2 \partial_n \left(\frac{n-1}{n} S_n\right) = -n^2 \partial_n \left(\frac{1}{n} \ln \mathrm{Tr}\rho_A^n\right)$$

- This is standard thermodynamic relation

$$\widetilde{S_n} = -\frac{\partial F_n}{\partial T}$$

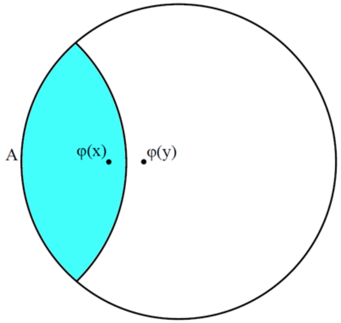
with $F_n = -\frac{1}{n} \ln \mathrm{Tr}\rho_A^n, \quad T = \frac{1}{n}$

$\mathrm{Tr}\rho_A^n$ is partition function w/ modular Hamiltonian $-\ln \rho_A$

- $\widetilde{S_n} \geq 0$ generally. [Beck & Schögl '93] Automatic by area law!

[XD]

JLMS relations



$$K = \frac{\widehat{\text{Area}}}{4G_N} + K_{\text{bulk}}$$

- CFT modular flow = bulk modular flow

$$S(\rho|\sigma) = S_{\text{bulk}}(\rho|\sigma)$$

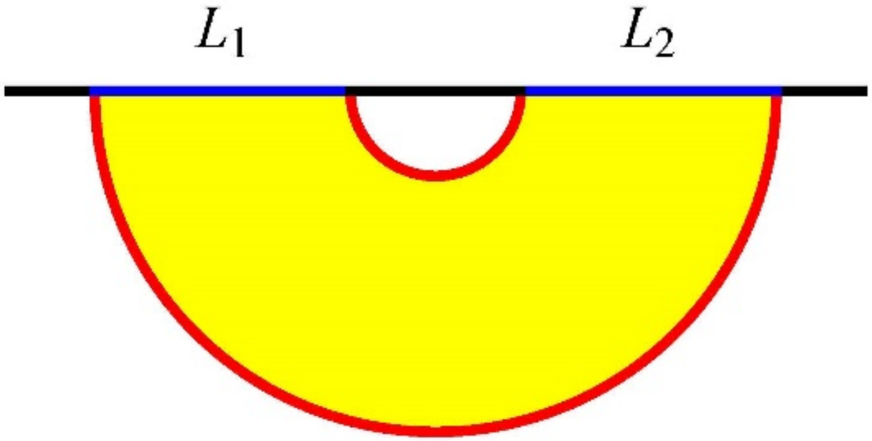
- Two states are as distinguishable in the CFT as in the bulk.
- Both relations hold to one-loop order in 1/N.

[Jafferis, Lewkowycz, Maldacena, Suh]

Plan

- Entanglement in QFT
- Entanglement in holography
- **Applications and generalizations**
- Holographic error correction

Phases of holographic entanglement entropy



- First-order phase transition
- Similar to Hawking-Page transition
- Mutual information $I(A,B) \stackrel{\text{def}}{=} S_A + S_B - S_{AB}$ changes from zero to nonzero (at order $1/G_N$).
- Holographic Renyi entropy has similar transitions.

Phases of holographic Renyi entropy

- Can have second-order phase transition at $n = n_{crit}$
- Need a sufficiently light bulk field
- Basic idea: increase $n \sim$ decrease T $\Rightarrow$ bulk field condenses.
- Examples:

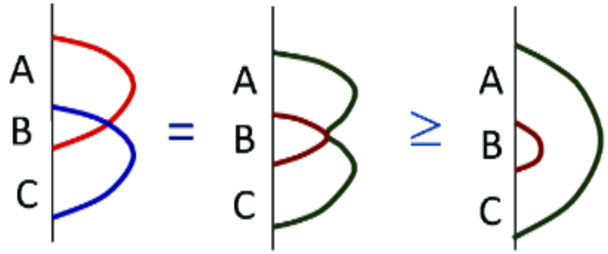
1. Spherical disk region
2. In $d = 2$: multiple intervals

[Belin, Maloney, Matsuura; XD, Maguire, Maloney, Maxfield]

Holographic Entropy Cone

- RT satisfies strong subadditivity:

[Headrick & Takayanagi]



$$\Rightarrow \quad S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_B$$

- Also satisfies other inequalities such as monogamy of mutual information

$$S_{AB} + S_{BC} + S_{CA} \geq S_{ABC} + S_A + S_B + S_C$$

[Hayden, Headrick & Maloney]

and

$$S_{ABC} + S_{BCD} + S_{CDE} + S_{DEA} + S_{EAB} \geq S_{ABCDE} + S_{AB} + S_{BC} + S_{CD} + S_{DE} + S_{EA}$$

[Bao, Nezami, Ooguri, Stoica, Sully & Walter]

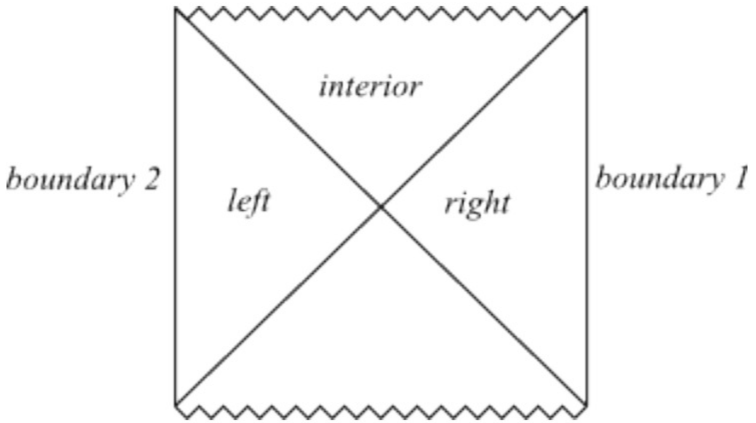
- Provide nontrivial conditions for a theory to have a gravity dual.
- Holographic entropy cone for time-dependent states?
- AdS_3/CFT_2: same as the static case. [XD & Czech, to appear]

Time evolution of entanglement

- Consider the thermofield double (TFD) state:

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{\beta E_n/2} |n\rangle_L |n\rangle_R$$

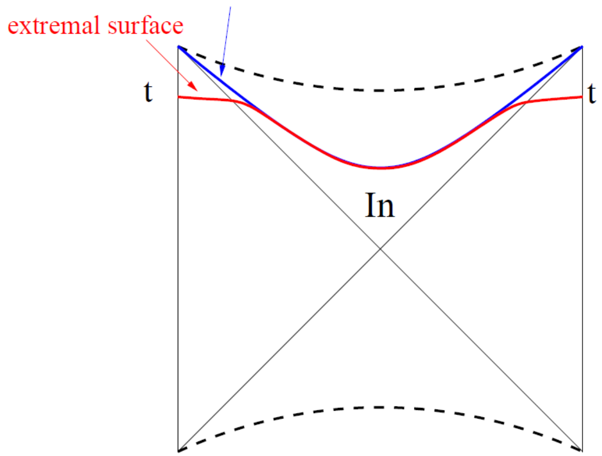
- Dual to eternal AdS black hole:



[Maldacena]

Time evolution of entanglement

- Consider the union of half of the left CFT and half of the right CFT at time t:



- Its entanglement entropy grows linearly in t.
- Related to the growth of the BH interior.

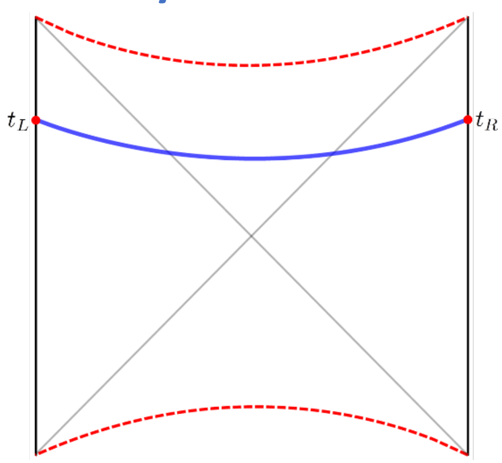
[Hartman, Maldacena]

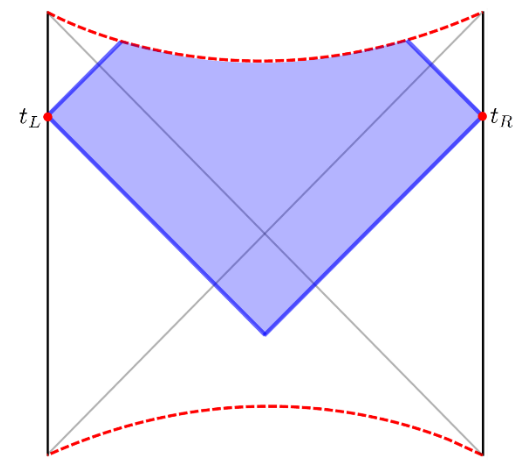
Holographic complexity

- Such growth of the BH interior is conjectured to correspond to the linear growth of complexity of the state.
- Complexity: minimum # of gates to prepare a state

➢ Complexity = Volume

➢ Complexity = Action

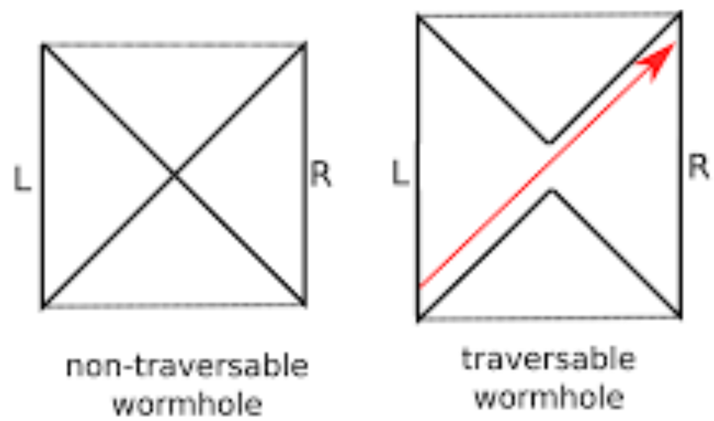




[Stanford, Susskind; Brown, Roberts, Susskind, Swingle, Zhao]

Traversable wormhole

- TFD is a highly entangled state, but the two CFTs do not interact with each other.
- Dual to a non-traversable wormhole.



- Can be made traversable by adding interactions.
- Quantum teleportation protocol.

[Gao, Jafferis, Wall; Maldacena, Stanford, Yang; Maldacena, Qi]

Einstein equations from entanglement

- The proof of RT relied on Einstein equations.
- Can go in the other direction: deriving Einstein equations from RT.
- Basic idea: first law of entanglement entropy

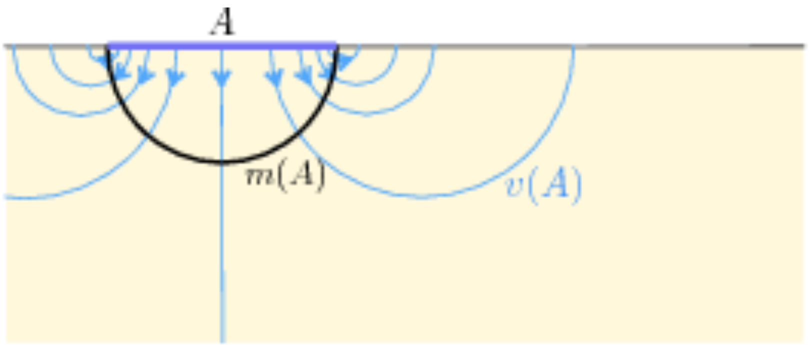
$$\delta S = \delta\langle K_\rho \rangle \quad \text{under} \quad \rho \to \rho + \delta\rho$$

- Apply to a spherical disk in the CFT vacuum, so that K_ρ is locally determined from the CFT stress tensor.
- Both sides are controlled by the bulk metric: via RT on the left, and via extrapolate dictionary on the right.
- Gives the (linearized) Einstein equations.

[XD, Faulkner, Guica, Haehl, Hartman, Hijano, Lashkari, Lewkowycz, McDermott, Myers, Parrikar, Rabideau, Swingle, VanRaamsdonk, ...]

Bit threads

- Reformulate RT using maximal flows instead of minimal surfaces.



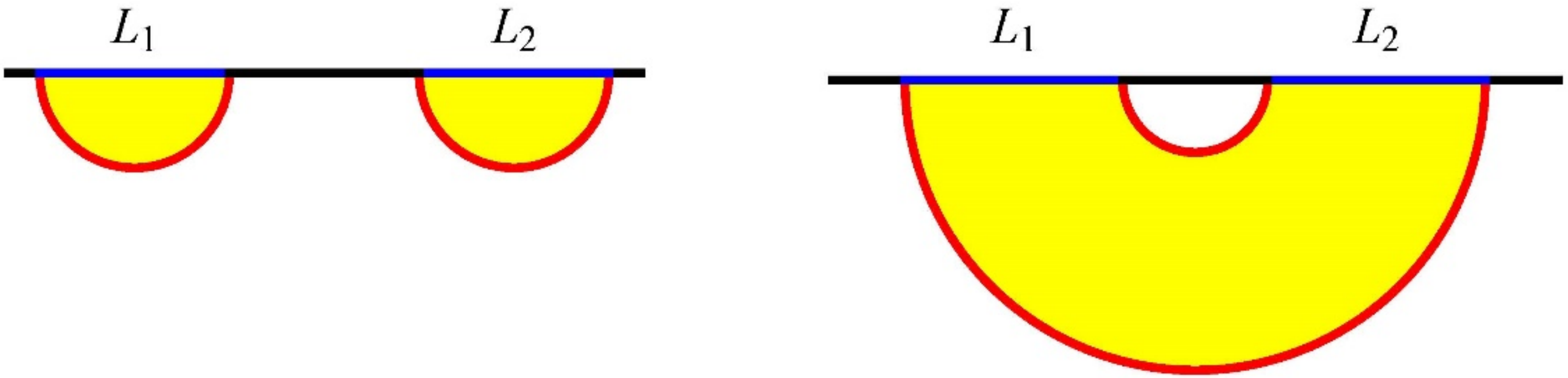
- Flow: $\nabla \cdot v = 0, \quad |v| \leq C$
- Max flow-min cut theorem:

$$\max_{v} \int_{A} v = C \min_{m \sim A} \text{Area}(m)$$

[Freedman, Headrick; Headrick, Hubeny]

Bit threads

- One advantage is that max flows change continuously across phase transitions, unlike min cuts:

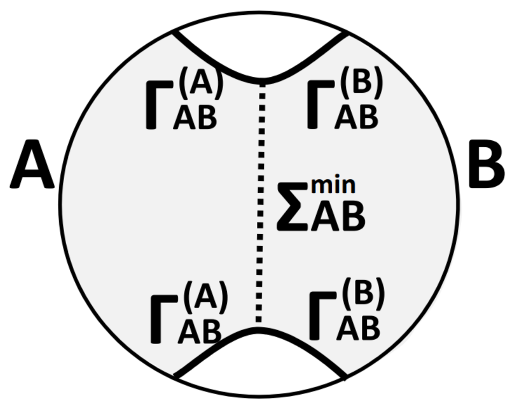


- This bit thread picture has been generalized to HRT.

[Freedman, Headrick; Headrick, Hubeny]

Entanglement of purification

- Given a mixed state ρ_{AB}, it may be purified as $|\psi\rangle_{AA'BB'}$.
- Entanglement of purification: $\min_{\psi} S(\rho_{AA'})$
- Conjectured to be holographically given by the entanglement wedge cross section $\Sigma_{AB}^{\min}$:



[Takayanagi, Umemoto, ...]

Modular flow as a disentangler

- "Modular minimal entropy" is given by the area of a constrained extremal surface.
- Given two regions R, A, define the modular evolved state: $|\psi(s)\rangle = \rho_R^{is}|\psi\rangle$
- And the modular minimal entropy:

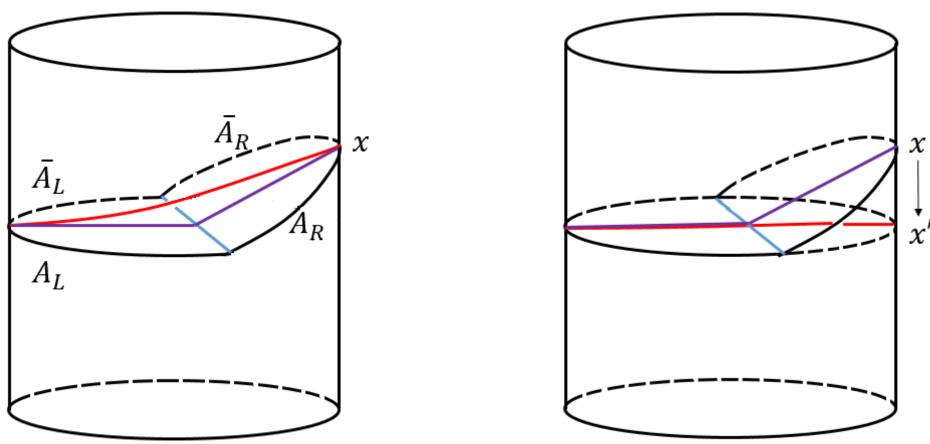
$$\overline{S_R}(A) = \min_s S_A(|\psi(s)\rangle)$$

- Holographically given by a constrained extremal surface (i.e. extremal except at intersections with the HRT surface γ_R of R).

[Chen, XD, Lewkowycz & Qi, appearing today]

Modular flow as a disentangler

- "Modular minimal entropy" is given by the area of a constrained extremal surface.
- Obvious when the modular flow is local:



- Seems to be generally true (can be proven in d=2).
- A useful diagnostic of whether γ_A and γ_R intersect.

[Chen, XD, Lewkowycz & Qi, appearing today]

Entanglement in de Sitter holography

- dS/dS correspondence:

$$ds^2_{dS_{d+1}} = dw^2 + \sin^2(w)\, ds^2_{dS_d}$$

- Suggests a dual description as two matter sectors coupled to each other and to d-dimensional gravity.
- Reminiscent of (but different from) the TFD example discussed previously.
- In particular, the two sides are in a maximally entangled state (to leading order), as shown by entanglement and Renyi entropies.
- Matches and gives an interpretation of the Gibbons-Hawking entropy.

[XD, Silverstein, Torroba]

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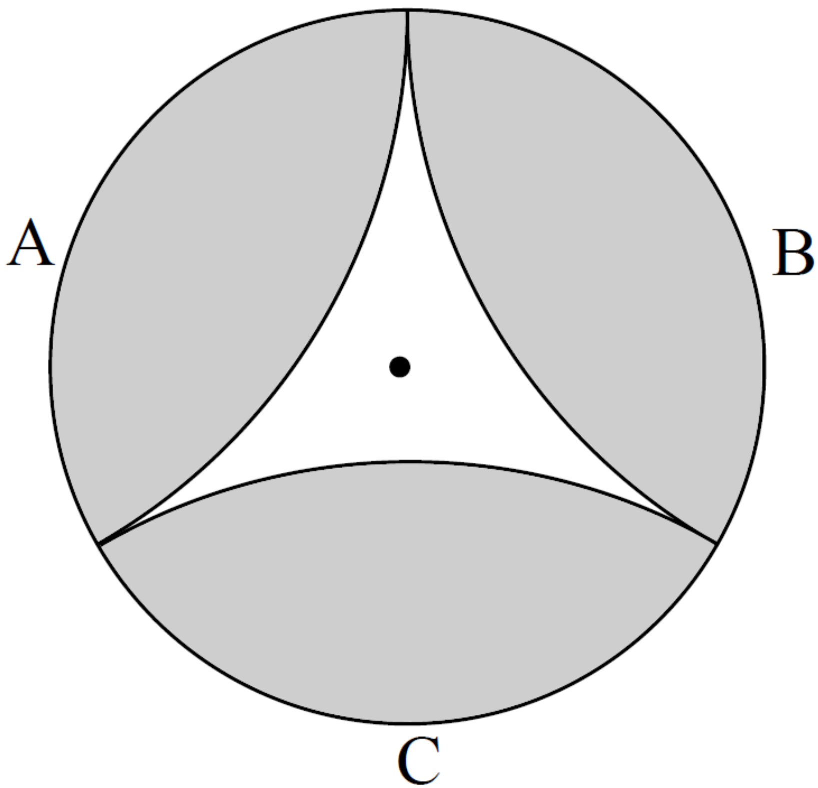
Why quantum error correction?



- $\phi(x)$ can be represented on $A \cup B$, $B \cup C$, or $A \cup C$.
- Obviously they cannot be the same CFT operator.
- Defining feature for quantum error correction.
- Holography is a quantum error correcting code.
- Reconstruction works in a code subspace of states.

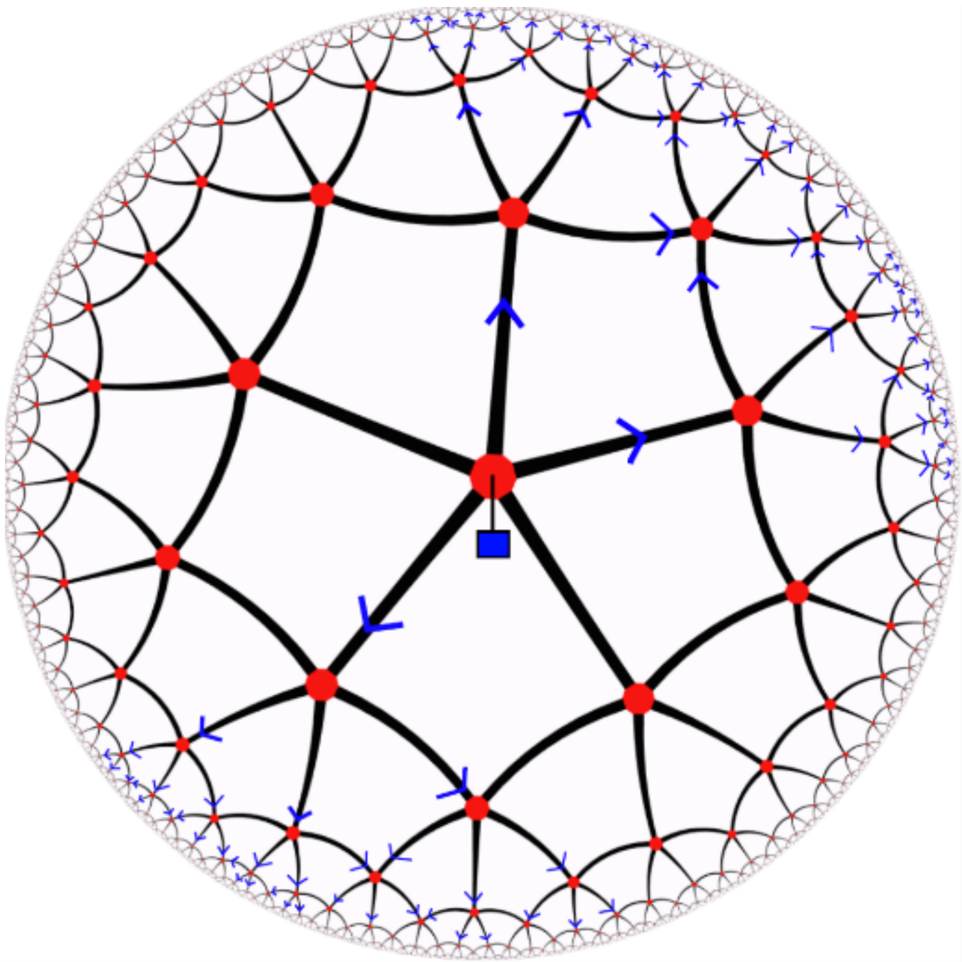
[Almheiri, XD, Harlow ’14]

AdS/CFT as a Quantum Error Correcting Code

Bulk Gravity	**Quantum Error Correction**
• Low-energy bulk states	• States in the code subspace
• Different CFT representations of a bulk operator	• Redundant implementation of the same logical operation
• Algebra of bulk operators	• Algebra of protected operators acting on the code subspace
• Radial distance	• Level of protection

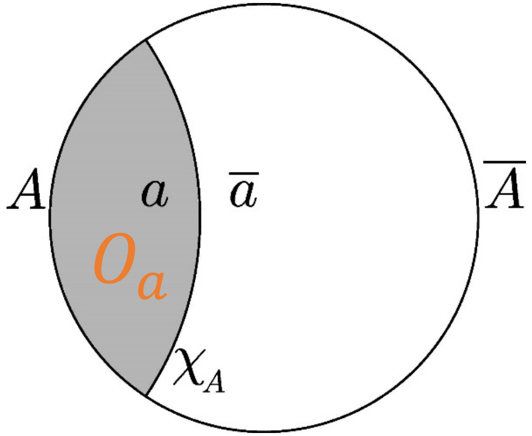
[Almheiri, XD, Harlow]

Tensor network toy models



[Pastawski, Yoshida, Harlow & Preskill '15; Hayden, Nezami, Qi, Thomas, Walter & Yang '16; Donnelly, Michel, Marolf & Wien '16; …]

Entanglement wedge reconstruction

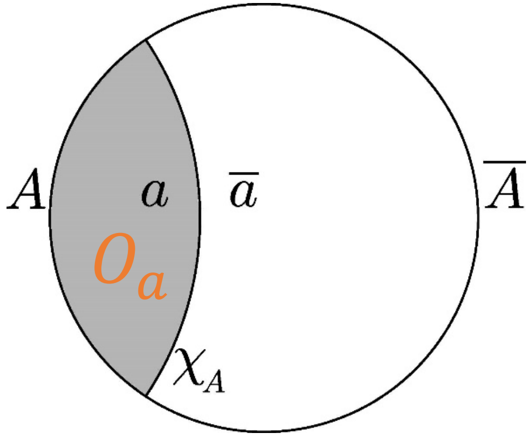


> Any bulk operator in the entanglement wedge of region A may be represented as a CFT operator on A.

In other words:

$\forall\ O_a$ in the entanglement wedge of A, $\exists\ O_A$ on A, s.t. $O_A|\phi\rangle = O_a|\phi\rangle$ and $O_A^\dagger|\phi\rangle = O_a^\dagger|\phi\rangle$ hold for $\forall\ |\phi\rangle \in H_{code}$.

Entanglement wedge reconstruction

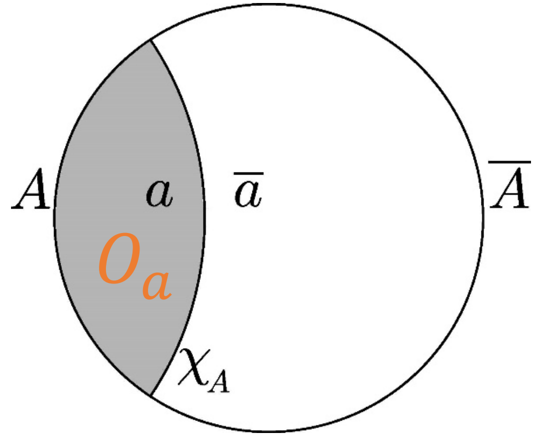


Any bulk operator in the entanglement wedge of region A may be represented as a CFT operator on A.

This new form of subregion duality goes beyond the old "causal wedge reconstruction": entanglement wedge can reach behind black hole horizons.

Valid to all orders in G_N.

Reconstruction theorem



- Goal is to prove: $\langle\phi|[O_a, X_{\bar{A}}]|\phi\rangle = 0$
- This is necessary and sufficient for

$$\exists\, O_A,\ s.t.\ O_A|\phi\rangle = O_a|\phi\rangle \text{ and } O_A^\dagger|\phi\rangle = O_a^\dagger|\phi\rangle$$

- WLG assume O_a is Hermitian.

[Almheiri, XD & Harlow '14]

- Consider two states $|\phi\rangle, e^{i\lambda O_a}|\phi\rangle$ in code subspace H_c:

$$S(\rho_{\bar{A}}|\sigma_{\bar{A}}) = S(\rho_{\bar{a}}|\sigma_a) = 0$$

⇓

$$\langle\phi|e^{-i\lambda O_a}X_{\bar{A}}e^{i\lambda O_a}|\phi\rangle = \langle\phi|X_{\bar{A}}|\phi\rangle$$

⇒

$$\langle\phi|[O_a, X_{\bar{A}}]|\phi\rangle = 0$$

[XD, Harlow & Wall 1601.05416]

RT from QEC

- We saw that entanglement wedge reconstruction arises from JLMS which is in turn a result of the quantum-corrected RT formula.
- Can go in the other direction.
- In any quantum error-correcting code with complementary recovery, we have a version of the RT formula with one-loop quantum corrections:

$$S_A = \langle A \rangle + S_a$$

[Harlow]

Conclusion and Questions

- Patterns of entanglement provide a unconventional window towards understanding QFT and gravity.
  - ➢ Can we generalize the entanglement-based proof of c-, F-, and a-theorems to higher dimensions?
- We discussed α' corrections to the RT formula.
  - ➢ Is there a stringy derivation that works for finite α'?
- We saw that RT with one-loop quantum corrections matches QEC with complementary recovery.
  - ➢ Do higher-order corrections mean something in QEC?
  - ➢ What conditions for a code to have a gravity dual?
  - ➢ How do we fully enjoy all of this in the context of black hole information problem and cosmology?

Thank you!